

# The homotopy type of spaces of resultants of bounded multiplicity

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## Abstract

For positive integers  $m, n, d \geq 1$  with  $(m, n) \neq (1, 1)$  and a field  $\mathbb{F}$  with its algebraic closure  $\overline{\mathbb{F}}$ , let  $\text{Poly}_n^{d,m}(\mathbb{F})$  denote the space of all  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in \mathbb{F}[z]^m$  of monic polynomials of the same degree  $d$  such that polynomials  $f_1(z), \dots, f_m(z)$  have no common root in  $\overline{\mathbb{F}}$  of multiplicity  $\geq n$ . These spaces were defined by Farb and Wolfson in [11] as generalizations of spaces first studied by Arnold, Vassiliev, Segal and others in different contexts. In [11] they obtained algebraic geometrical and arithmetic results about the topology of these spaces. In this paper we investigate the homotopy type of these spaces for the case  $\mathbb{F} = \mathbb{C}$ . Our results generalize those of [11] for  $\mathbb{F} = \mathbb{C}$  and also results of G. Segal [29], V. Vassiliev [31] and F.Cohen-R.Cohen-B.Mann-R.Milgram [5] for  $m \geq 2$  and  $n \geq 2$ .

## 1 Introduction

**1.1 Historical survey.** There are two related intriguing phenomena that have been observed in various situations and can be roughly described as follows. Let  $X$  and  $Y$  be two manifolds with some structure (e.g. holomorphic, symplectic, real algebraic) and let  $\{M_d\}$  be a family of subspaces of structure-preserving continuous mappings  $X \rightarrow Y$  indexed by “degree” (for a suitable notion of index) with “stabilization mappings”  $q_d : M_d \rightarrow M_{d+1}$ . The first phenomenon is that the homology (homotopy) groups of the subspaces  $M_d$  stabilize, that is: the mappings  $q_d$  are homology (homotopy) equivalences

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2010 *Mathematics Subject Classification.* Primary 55P10; Secondly 55R80, 55P35.

up to some dimension  $n(d)$  which is a monotonically increasing function of  $d$ . The other phenomenon is that a certain limit of the subspaces  $M_d$  is homology (homotopy) equivalent to the space of all continuous mappings  $X \rightarrow Y$ .

It seems that the first appearance of such phenomena was in the work of V. I. Arnold [2]. Arnold considered the space  $\mathrm{SP}_n^d(\mathbb{C})$  of complex monic polynomials of the degree  $d$  without roots of multiplicity  $n$ . For the case  $n = 2$  this is the same as the space of monic polynomials without repeated roots (with non-zero discriminant), whose fundamental group is the braid group  $\mathrm{Br}(d)$  of  $d$ -strings and whose cohomology is the cohomology of the braid group  $\mathrm{Br}(d)$ . Arnold computed the homology of these braid groups and established their homological stability. The corresponding relationship between these spaces of polynomials and spaces of continuous maps is given by the May-Segal theorem [28]. Similar results also hold in the real polynomial case ([16], [20], [31]).

Analogous another phenomena were discovered by G. Segal [29] in a different context inspired by control theory (later it was discovered to have a close relationship with mathematical physics [3]). Segal considered the space  $\mathrm{Hol}_d^*(\mathbb{CP}^1, \mathbb{CP}^{n-1})$  of based holomorphic maps from  $\mathbb{CP}^1$  to  $\mathbb{CP}^{n-1}$  of the degree  $d$  and its inclusion into the space  $\mathrm{Map}_d^*(\mathbb{CP}^1, \mathbb{CP}^{n-1}) = \Omega_d^2 \mathbb{CP}^{n-1}$  of corresponding continuous maps. Intuitive considerations based on Morse theory, suggest that homotopy of the first space should approximate that of the second space more and more closely as the degree  $d$  increases. Segal proved this result by observing that this space of based holomorphic mappings can be identified with the space of  $n$ -tuples  $(f_1(z), \dots, f_n(z)) \in \mathbb{C}[z]^n$  of monic polynomials of the same degree  $d$  without common roots. He defined a stabilization map  $\mathrm{Hol}_d^*(\mathbb{CP}^1, \mathbb{CP}^{n-1}) \rightarrow \mathrm{Hol}_{d+1}^*(\mathbb{CP}^1, \mathbb{CP}^{n-1})$  and proved that the induced maps on homotopy groups are isomorphisms up to some dimension increasing with  $d$ . Using a different technique (based on ideas of Gromov and Dold-Thom) he also proved that there is a homotopy equivalence  $q : \varinjlim \mathrm{Hol}_d^*(\mathbb{CP}^1, \mathbb{CP}^{n-1}) \xrightarrow{\simeq} \Omega_0^2 \mathbb{CP}^{n-1}$  defined by a “scanning of particles”, and that this equivalence is homotopic to the inclusion of the space of all holomorphic maps into the space of all continuous maps.

With the help of the spectral sequence for complements of discriminants (analogous to the one he used in defining invariants of knots) Vassilev [31] showed that there is a stable homotopy equivalence

$$(1.1) \quad \mathrm{SP}_n^d(\mathbb{C}) \simeq_s \mathrm{Hol}_{\lfloor \frac{d}{n} \rfloor}^*(\mathbb{CP}^1, \mathbb{CP}^{n-1}),$$

where  $\lfloor x \rfloor$  denotes the integer part of a real number  $x$ .

The relationship between Arnold's and Segal's arguments can also be explained in terms of Gromov's h-principle in [16] (cf. [14]), and in [17] it was shown that there is a homotopy equivalence<sup>1</sup>

$$(1.2) \quad \mathrm{SP}_n^d(\mathbb{C}) \simeq \mathrm{Hol}_{[\frac{d}{n}]}^*(\mathbb{CP}^1, \mathbb{CP}^{n-1}), \quad \text{if } n \geq 3$$

(see Theorem 1.7). The argument makes use of the existence of a  $C_2$ -operad actions on the spaces  $\coprod_{d \geq 0} \mathrm{SP}_n^d(\mathbb{C})$  and  $\coprod_{d \geq 0} \mathrm{Hol}_d^*(\mathbb{CP}^1, \mathbb{CP}^{n-1})$  ([4], [9], [17]).

Recently Benson Farb and Jesse Wolfson [11] made a remarkable discovery. In order to state their results in full generality<sup>2</sup> Farb and Wolfson defined a new algebraic variety, given in terms of  $m$ -tuples of monic polynomials with conditions on common roots. Namely, for a field  $\mathbb{F}$  with its algebraic closure  $\overline{\mathbb{F}}$ , let  $\mathrm{Poly}_n^{d,m}(\mathbb{F})$  denote the space of  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in \mathbb{F}[z]^m$  of monic polynomials of the same degree  $d$  with no common root in  $\overline{\mathbb{F}}$  of multiplicity  $n$  or greater.

For example, if  $\mathbb{F} = \mathbb{C}$ ,  $\mathrm{Poly}_n^{d,1}(\mathbb{C}) = \mathrm{SP}_n^d(\mathbb{C})$  and  $\mathrm{Poly}_1^{d,n}(\mathbb{C})$  can be identified with the space  $\mathrm{Hol}_d^*(\mathbb{CP}^1, \mathbb{CP}^{n-1})$ . Note that in terms of this definition the homotopy equivalence (1.2) can be expressed as the homotopy equivalence

$$(1.3) \quad \mathrm{Poly}_n^{d,1}(\mathbb{C}) \simeq \mathrm{Poly}_1^{[\frac{d}{n}],n}(\mathbb{C}) \quad \text{if } n \geq 3.$$

By the classical theory of resultants, the space  $\mathrm{Poly}_n^{d,m}(\mathbb{C})$  is an affine variety defined by systems of polynomial equations with integer coefficients. Thus both varieties given in (1.3) can be defined over  $\mathbb{Z}$  and (by extension of scalars or reduction modulo a prime number) over any field  $\mathbb{F}$ . Farb and Wolfson computed various algebraic and arithmetic invariants (such as the number of points for a finite field  $\mathbb{F}_q$ , etale cohomology etc) of these varieties and found that these invariants are always equal. They asked the natural question: are these varieties algebraically isomorphic for  $n \geq 3$ ?

In the simplest case  $(d, n) = (3, 3)$ , Curt MacMullen constructed an isomorphism between  $\mathrm{Poly}_3^{3,1}(\mathbb{Z}[\frac{1}{3}])$  and  $\mathrm{Poly}_1^{1,3}(\mathbb{Z}[\frac{1}{3}])$  and a different isomorphism between  $\mathrm{Poly}_3^{3,1}(\mathbb{Z}/3)$  and  $\mathrm{Poly}_1^{1,3}(\mathbb{Z}/3)$  ([11]). The formula defining the first of these isomorphisms of course gives also an isomorphism over  $\mathbb{C}$  and  $\mathbb{R}$ , and hence, of course, implies that these spaces are homeomorphic and thus homotopy equivalent. This example suggests that it is unlikely that the varieties  $\mathrm{Poly}_n^{dn,1}(\mathbb{Z})$  and  $\mathrm{Poly}_1^{d,n}(\mathbb{Z})$  are isomorphic but are more likely to be

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<sup>1</sup>Since  $\pi_1(\mathrm{SP}_2^d(\mathbb{C})) = \mathrm{Br}(d)$ , it is not commutative. However,  $\pi_1(\mathrm{Hol}_{[\frac{d}{n}]}^*(\mathbb{CP}^1, \mathbb{CP}^1)) = \mathbb{Z}$ . So two spaces  $\mathrm{SP}_n^d(\mathbb{C})$  and  $\mathrm{Hol}_{[\frac{d}{n}]}^*(\mathbb{CP}^1, \mathbb{CP}^{n-1})$  are not homotopy equivalent if  $n = 2$ .

<sup>2</sup>They have been generalized even farther in [12].

so if we invert the primes dividing  $d$ . As before, such an isomorphism of varieties (over a local ring) induces an isomorphism over both  $\mathbb{C}$  and  $\mathbb{R}$ . The question posed by Farb and Wolfson seems difficult to answer, and doing so would certainly require completely different methods from the ones used here. However, our results will lead to its generalization. Namely, in this paper we will show that there is a homotopy equivalence

$$(1.4) \quad \text{Poly}_n^{d,m}(\mathbb{C}) \simeq \text{Poly}_1^{\lfloor \frac{d}{n} \rfloor, mn}(\mathbb{C}) \quad \text{if } mn \geq 3$$

(see Theorem 1.11). This naturally leads to the question of whether this homotopy equivalence derives from an isomorphism of varieties. Such an isomorphism should be defined for varieties over  $\mathbb{Q}$  and perhaps over the ring  $\mathbb{Z}[S^{-1}]$  where  $S$  is some set of primes. Of course if this is the case, the same must hold with coefficients in  $\mathbb{R}$ . We should therefore expect to be able to prove homotopy equivalence in the real case too. We intend to pursue this topic in the future work [24].

We would like to note here the key role played in our argument by the “jet map” (see (1.5) below). The jet map is an algebraic map and can be defined over any field  $\mathbb{F}$ . Over  $\mathbb{C}$  it induces a homotopy equivalence of suitable stabilizations (Theorem 4.6) but it is not a candidate for the conjectured isomorphism since it does not have the right target space.

It is interesting to observe that there is another “real analogue” of (1.3) given in [20]. It is obtained by having on the left hand side the space of real polynomials without real roots of multiplicity  $\geq n$  (considered by Arnold and Vassiliev [31]) and on the right hand side the space of real rational maps from  $\mathbb{RP}^1$  to  $\mathbb{RP}^{n-1}$  in the sense of Mostovoy [26], which can be identified with the space represented by  $n$ -tuples of real coefficients monic polynomials of the same degree without common real roots (but possibly with common complex roots). Since these spaces are semi-algebraic varieties rather than algebraic varieties, the above argument does not seem to apply, but intriguingly the analogous result of (1.4) remains true. This will be proved in another work [23].

The purpose of this article is to determine the homotopy type of the space  $\text{Poly}_n^{d,m}(\mathbb{F})$  for  $\mathbb{F} = \mathbb{C}$  which, from now on, will be denoted simply as  $\text{Poly}_n^{d,m} = \text{Poly}_n^{d,m}(\mathbb{C})$  (see Definition 1.2). The homotopy equivalence (1.4) is an immediate consequence of this (Theorem 1.11). Our arguments are generally analogous to those in [16], [17] and [21], but the technical details are more complicated.

**1.2 Basic definitions and notations.** For connected spaces  $X$  and  $Y$ , let  $\text{Map}(X, Y)$  (resp.  $\text{Map}^*(X, Y)$ ) denote the space consisting of all continuous maps (resp. base-point preserving continuous maps) from  $X$  to  $Y$  with the compact-open topology. When  $X$  and  $Y$  are complex manifolds, we denote by  $\text{Hol}(X, Y)$  (resp.  $\text{Hol}^*(X, Y)$ ) the subspace of  $\text{Map}(X, Y)$  (resp.  $\text{Map}^*(X, Y)$ ) consisting of all holomorphic maps (resp. base-point preserving holomorphic maps).

For each integer  $d \geq 0$ , let  $\text{Map}_d^*(S^2, \mathbb{CP}^{N-1}) = \Omega_d^2 \mathbb{CP}^{N-1}$  denote the space of all based continuous maps  $f : (S^2, \infty) \rightarrow (\mathbb{CP}^{N-1}, *)$  such that  $[f] = d \in \mathbb{Z} = \pi_2(\mathbb{CP}^{N-1})$ , where we identify  $\mathbb{CP}^1 = S^2 = \mathbb{C} \cup \{\infty\}$  and the points  $\infty \in S^2$  and  $*$   $\in \mathbb{CP}^{N-1}$  are the base points of  $S^2$  and  $\mathbb{CP}^{N-1}$ , respectively. Let  $\text{Hol}_d^*(S^2, \mathbb{CP}^{N-1})$  denote the subspace of  $\text{Map}_d^*(S^2, \mathbb{CP}^{N-1})$  consisting of all based holomorphic maps of degree  $d$ .

**Remark 1.1.** Let  $\text{P}^d(\mathbb{C})$  denote the space consisting of all complex monic polynomials  $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{C}[z]$  of the degree  $d$ .

If we choose the point  $[1 : 1 : \cdots : 1] \in \mathbb{CP}^{N-1}$  as its base point, the space  $\text{Hol}_d^*(S^2, \mathbb{CP}^{N-1})$  can be identified with the space consisting of all  $N$ -tuples  $(f_1(z), \dots, f_N(z)) \in \text{P}^d(\mathbb{C})^N$  of monic polynomials of the same degree  $d$  such that polynomials  $f_1(z), \dots, f_N(z)$  have no common root, i.e. the space  $\text{Hol}_d^*(S^2, \mathbb{CP}^{N-1})$  can be identified with

$$\text{Hol}_d^*(S^2, \mathbb{CP}^{N-1}) = \{(f_1, \dots, f_N) \in \text{P}^d(\mathbb{C})^N : \{f_k(z)\}_{k=1}^N \text{ have no common root}\}.$$

**Definition 1.2.** (i) Let  $\text{SP}_n^d(\mathbb{C})$  denote the space of all monic polynomials  $f(z) \in \text{P}^d(\mathbb{C})$  of the degree  $d$  without root of multiplicity  $\geq n$ . More generally, for positive integers  $m, n, d \geq 1$  with  $(m, n) \neq (1, 1)$ , let  $\text{Poly}_n^{d,m}$  denote the space of all  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in (\text{P}^d(\mathbb{C}))^m$  of monic polynomials of the same degree  $d$  such that polynomials  $f_1(z), \dots, f_m(z)$  have no common root of multiplicity  $\geq n$ .

(ii) Let  $(f_1(z), \dots, f_m(z)) \in \text{P}^d(\mathbb{C})^m$ . Note that  $(f_1(z), \dots, f_m(z)) \in \text{Poly}_n^{d,m}$  iff the polynomials  $\{f_j^{(k)}(z) : 1 \leq j \leq m, 0 \leq k < n\}$  have no common root. In this situation, define the *jet map*

$$(1.5) \quad j_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}) \quad \text{by}$$

$$(1.6) \quad j_n^{d,m}(f_1(z), \dots, f_m(z)) = (\mathbf{f}_1(z), \dots, \mathbf{f}_m(z))$$

for  $(f_1(z), \dots, f_m(z)) \in \text{Poly}_n^{d,m}$ , where  $\mathbf{f}_k(z)$  ( $k = 1, 2, \dots, m$ ) denotes the  $n$ -tuple of monic polynomials of the same degree  $d$  defined by

$$(1.7) \quad \mathbf{f}_k(z) = (f_k(z), f_k(z) + f_k'(z), f_k(z) + f_k''(z), \dots, f_k(z) + f_k^{(n-1)}(z)).$$

Let  $i_n^{d,m}$  denote the natural map  $i_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \Omega_d^2 \mathbb{CP}^{mn-1} \simeq \Omega^2 S^{2mn-1}$  defined by

$$(1.8) \quad i_n^{d,m}(f_1(z), \dots, f_m(z))(\alpha) = \begin{cases} [\mathbf{f}_1(\alpha) : \dots : \mathbf{f}_m(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1 : 1 : \dots : 1] & \text{if } \alpha = \infty \end{cases}$$

for  $(f_1(z), \dots, f_m(z)) \in \text{Poly}_n^{d,m}$  and  $\alpha \in S^2 = \mathbb{C} \cup \infty$ .  $\square$

**Remark 1.3.** (i) Note that  $\text{SP}_n^d(\mathbb{C}) = \text{Poly}_n^{d,1}$  for  $m = 1$ , and that we can identify  $\text{Poly}_1^{d,m} = \text{Hol}_d^*(S^2, \mathbb{CP}^{m-1})$  for  $n = 1$ . It is easy to see that there is a homeomorphism

$$(1.9) \quad \text{Poly}_n^{d,m} \cong \begin{cases} \mathbb{C}^{dm} & \text{if } d < n, \\ \mathbb{C}^{mn} \setminus \{(x, \dots, x) : x \in \mathbb{C}\} & \text{if } d = n. \end{cases}$$

Thus in this paper we shall mainly consider the case  $m \geq 2$  with  $d \geq n \geq 2$ .

(ii) A map  $f : X \rightarrow Y$  is called a *homotopy equivalence* (resp. a *homology equivalence*) *through dimension  $D$*  if the induced homomorphism  $f_* : \pi_k(X) \rightarrow \pi_k(Y)$  (resp.  $f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z})$ ) is an isomorphism for any  $k \leq D$ .  $\square$

**1.3 The related known results.** Now, recall the following known results. First, consider the case  $m = 1$ . Note that  $\text{Poly}_n^{d,1} = \text{SP}_n^d(\mathbb{C})$ .

**Theorem 1.4** ([16], [22]). *The jet map*

$$j_n^{d,1} : \text{SP}_n^d(\mathbb{C}) \rightarrow \Omega_d^2 \mathbb{CP}^{n-1}$$

*is a homotopy equivalence through dimension  $(2n-3)(\lfloor \frac{d}{n} \rfloor + 1) - 1$  if  $n \geq 3$  and it is a homology equivalence thorough dimension  $\lfloor \frac{d}{2} \rfloor$  if  $n = 2$ .*  $\square$

Next, consider the case  $m \geq 2$  and  $n = 1$ . In this case, we can identify  $\text{Poly}_1^{d,m} = \text{Hol}_d^*(S^2, \mathbb{CP}^{m-1})$  and the following result is known.

**Theorem 1.5** ([21], [29]). *If  $m \geq 2$ , the inclusion map*

$$i_d = j_1^{d,m} : \text{Hol}_d^*(S^2, \mathbb{CP}^{m-1}) \rightarrow \Omega_d^2 \mathbb{CP}^{m-1} \simeq \Omega^2 S^{2m-1}$$

*is a homotopy equivalence through dimension  $(2m-3)(d+1) - 1$ .*  $\square$

We also recall the stable result obtained by F.Cohen-R.Cohen-B.Mann-R.Milgram and its improvement due to R.Cohen-D.Shimamoto ([5], [6], [9]).

**Theorem 1.6** ([5], [6], [9]). (i) If  $m \geq 2$ , there is a stable homotopy equivalence

$$\mathrm{Hol}_d^*(S^2, \mathbb{CP}^{m-1}) \simeq_s \bigvee_{k=1}^d \Sigma^{2(m-2)k} D_k,$$

where  $\Sigma^k X$  denotes the  $k$ -fold reduced suspension of a based space  $X$  and  $D_k$  is the equivariant half smash product  $D_k = F(\mathbb{C}, k)_+ \wedge_{S_k} (\wedge^k S^1)$  defined by (3.10).

(ii) In particular, if  $m \geq 3$ , there is a homotopy equivalence

$$\mathrm{Hol}_d^*(S^2, \mathbb{CP}^{m-1}) \simeq J_2(S^{2m-3})_d,$$

where  $J_2(S^{2m-3})_d$  denotes the  $d$ -th stage filtration of the May-Milgram model for  $\Omega^2 S^{2m-1}$  defined by (6.12).  $\square$

Note that the homotopy types of the spaces  $\mathrm{SP}_n^d(\mathbb{C})$  and  $\mathrm{Hol}_{[\frac{d}{n}]}^*(S^2, \mathbb{CP}^{n-1})$  are closely connected in the following sense.

**Theorem 1.7** ([17], [31]). (i) If  $n \geq 3$ , there is a homotopy equivalence<sup>3</sup>

$$\mathrm{SP}_n^d(\mathbb{C}) \simeq \mathrm{Hol}_{[\frac{d}{n}]}^*(S^2, \mathbb{CP}^{n-1}).$$

(ii) If  $n = 2$ , there is a stable homotopy equivalence

$$\mathrm{SP}_2^d(\mathbb{C}) \simeq_s \mathrm{Hol}_{[\frac{d}{2}]}^*(S^2, \mathbb{CP}^1). \quad \square$$

**1.4 The main results.** The main purpose of this paper is to investigate the homotopy type of the space  $\mathrm{Poly}_n^{d,m}$  and generalize the above results. Since the case  $m = 1$  or the case  $n = 1$  was already well studied in the above theorems, we mainly consider the case  $m \geq 2$  and  $n \geq 2$ . Let  $D(d; m, n)$  denote the positive integer defined by

$$(1.10) \quad D(d; m, n) = (2mn - 3) \left( \left\lfloor \frac{d}{n} \right\rfloor + 1 \right) - 1.$$

The main results of this paper is stated as follows.

**Theorem 1.8.** Let  $m$  and  $n$  be positive integers. If  $mn \geq 3$ , the natural map

$$i_n^{d,m} : \mathrm{Poly}_n^{d,m} \rightarrow \Omega_d^2 \mathbb{CP}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a homotopy equivalence through dimension  $D(d; m, n)$ .

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<sup>3</sup>This result was also proved independently by S. Kallel.

**Corollary 1.9.** *Let  $m$  and  $n$  be positive integers. If  $mn \geq 3$ , the jet map*

$$j_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})$$

*is a homotopy equivalence through dimension  $D(d; m, n)$ .*

By using the above result and the result obtained by R. Cohen and D. Shimamoto [9], we also obtain the following two results.

**Theorem 1.10.** *If  $m$  and  $n$  are positive integers with  $(m, n) \neq (1, 1)$ , there is a stable homotopy equivalence*

$$\text{Poly}_n^{d,m} \simeq_s \bigvee_{k=1}^{\lfloor \frac{d}{n} \rfloor} \Sigma^{2(mn-2)k} D_k.$$

**Theorem 1.11.** *Let  $m$  and  $n$  be positive integers. If  $mn \geq 3$ , then there is a homotopy equivalence*

$$\text{Poly}_n^{d,m} \simeq \text{Hol}_{\lfloor \frac{d}{n} \rfloor}^*(S^2, \mathbb{CP}^{mn-1}).$$

**Remark 1.12.** It is easy to see that the above homotopy equivalence given in Theorem 1.11 can also be expressed in the form (1.4).  $\square$

This paper is organized as follows. In §2 we recall the simplicial resolutions, and in §3 we shall construct the Vassiliev type spectral sequences converging to the homologies of the spaces  $\text{Poly}_n^{d,m}$  and  $\text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})$ . In §4, we consider the stabilization map  $s_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Poly}_n^{d+1,m}$  and prove the homological stability of the map  $s_n^{d,m}$  (Theorem 4.3). In §5 we prove Theorem 4.6 by using the scanning maps and we give the proofs of the main results of this paper (Theorem 1.8 and Corollary 1.9). In §6 we prove Theorem 1.10 and Theorem 1.11 by using Theorem 6.8. In §7 we give the proof of Theorem 6.8 by using Lemma 7.3, and in §8 we prove Lemma 7.3.

## 2 Simplicial resolutions

In this section, we give the definitions of and summarize the basic facts about non-degenerate simplicial resolutions ([31], [32], (cf. [27])).

**Definition 2.1.** (i) For a finite set  $\mathbf{v} = \{v_1, \dots, v_l\} \subset \mathbb{R}^N$ , let  $\sigma(\mathbf{v})$  denote the convex hull spanned by  $\mathbf{v}$ . Let  $h : X \rightarrow Y$  be a surjective map such that  $h^{-1}(y)$  is a finite set for any  $y \in Y$ , and let  $i : X \rightarrow \mathbb{R}^N$  be an embedding. Let  $\mathcal{X}^\Delta$  and  $h^\Delta : \mathcal{X}^\Delta \rightarrow Y$  denote the space and the map defined by

$$(2.1) \quad \mathcal{X}^\Delta = \{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\} \subset Y \times \mathbb{R}^N, \quad h^\Delta(y, u) = y.$$



The pair  $(\mathcal{X}^\Delta, h^\Delta)$  is called *the simplicial resolution of  $(h, i)$* . In particular,  $(\mathcal{X}^\Delta, h^\Delta)$  is called *a non-degenerate simplicial resolution* if for each  $y \in Y$  any  $k$  points of  $i(h^{-1}(y))$  span  $(k - 1)$ -dimensional simplex of  $\mathbb{R}^N$ .

(ii) For each  $k \geq 0$ , let  $\mathcal{X}_k^\Delta \subset \mathcal{X}^\Delta$  be the subspace given by

$$(2.2) \quad \mathcal{X}_k^\Delta = \{(y, u) \in \mathcal{X}^\Delta : u \in \sigma(\mathbf{v}), \mathbf{v} = \{v_1, \dots, v_l\} \subset i(h^{-1}(y)), l \leq k\}.$$

We make identification  $X = \mathcal{X}_1^\Delta$  by identifying  $x \in X$  with  $(h(x), i(x)) \in \mathcal{X}_1^\Delta$ , and we note that there is an increasing filtration

$$\emptyset = \mathcal{X}_0^\Delta \subset X = \mathcal{X}_1^\Delta \subset \mathcal{X}_2^\Delta \subset \dots \subset \mathcal{X}_k^\Delta \subset \mathcal{X}_{k+1}^\Delta \subset \dots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_k^\Delta = \mathcal{X}^\Delta.$$

Since the map  $h^\Delta$  is a proper map, it extends the map  $h_+^\Delta : \mathcal{X}_+^\Delta \rightarrow Y_+$  between one-point compactifications, where  $X_+$  denotes the one-point compactification of a locally compact space  $X$ .

**Theorem 2.2** ([31], [32] (cf. [22], [27])). *Let  $h : X \rightarrow Y$  be a surjective map such that  $h^{-1}(y)$  is a finite set for any  $y \in Y$ ,  $i : X \rightarrow \mathbb{R}^N$  an embedding, and let  $(\mathcal{X}^\Delta, h^\Delta)$  denote the simplicial resolution of  $(h, i)$ .*

- (i) *If  $X$  and  $Y$  are semi-algebraic spaces and the two maps  $h, i$  are semi-algebraic maps, then  $h_+^\Delta : \mathcal{X}_+^\Delta \xrightarrow{\sim} Y_+$  is a homology equivalence.<sup>4</sup> Moreover, there is an embedding  $j : X \rightarrow \mathbb{R}^M$  such that the associated simplicial resolution  $(\tilde{\mathcal{X}}^\Delta, \tilde{h}^\Delta)$  of  $(h, j)$  is non-degenerate.*
- (ii) *If there is an embedding  $j : X \rightarrow \mathbb{R}^M$  such that its associated simplicial resolution  $(\tilde{\mathcal{X}}^\Delta, \tilde{h}^\Delta)$  is non-degenerate, the space  $\tilde{\mathcal{X}}^\Delta$  is uniquely determined up to homeomorphism and there is a filtration preserving homotopy equivalence  $q^\Delta : \tilde{\mathcal{X}}^\Delta \xrightarrow{\sim} \mathcal{X}^\Delta$  such that  $q^\Delta|_X = \text{id}_X$ .  $\square$*

### 3 The Vassiliev spectral sequences

Our goal in this section is to construct, by means of the *non-degenerate* simplicial resolutions of the discriminants, two Vassiliev type spectral sequences converging to the homology of  $\text{Poly}_n^{d,m}$  and that of  $\text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})$ , respectively.

---

<sup>4</sup>It is known that the map  $h_+^\Delta$  is a homotopy equivalence [32, page 156]. (cf. [13, Theorem in page 43]). But in this paper we do not need such a stronger assertion.

**Definition 3.1.** (i) Let  $\Sigma_n^{d,m}$  denote *the discriminant* of  $\text{Poly}_n^{d,m}$  in  $\mathbb{P}^d(\mathbb{C})^m$  given by the complement

$$\begin{aligned}\Sigma_n^{d,m} &= \mathbb{P}^d(\mathbb{C})^m \setminus \text{Poly}_n^{d,m} \\ &= \{(f_1, \dots, f_m) \in \mathbb{P}^d(\mathbb{C})^m : \mathbf{f}_1(x) = \dots = \mathbf{f}_m(x) = \mathbf{0} \text{ for some } x \in \mathbb{C}\}.\end{aligned}$$

Let us write  $\mathbb{P}^d(m, n) = (\mathbb{P}^d(\mathbb{C})^n)^m$ , and let  $\tilde{\Sigma}^d$  be *the discriminant* of  $\text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})$  in  $\mathbb{P}^d(m, n)$  given by

$$\begin{aligned}\tilde{\Sigma}^d &= \mathbb{P}^d(m, n) \setminus \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}) \\ &= \{(f_1, \dots, f_{mn}) \in \mathbb{P}^d(m, n) : f_1(x) = \dots = f_{mn}(x) = 0 \text{ for some } x \in \mathbb{C}\}.\end{aligned}$$

(ii) Let  $Z_n^{d,m} \subset \Sigma_n^{d,m} \times \mathbb{C}$  denote *the tautological normalization* of  $\Sigma_n^{d,m}$  given by

$$Z_n^{d,m} = \{((f_1(z), \dots, f_m(z)), x) \in \Sigma_n^{d,m} \times \mathbb{C} : \mathbf{f}_1(x) = \dots = \mathbf{f}_m(x) = \mathbf{0}\}.$$

Similarly, let  $\tilde{Z}_N^d \subset \tilde{\Sigma}^d \times \mathbb{C}$  be *the tautological normalization* of  $\tilde{\Sigma}^d$  given by

$$\tilde{Z}^d = \{((f_1, \dots, f_{mn}), x) \in \tilde{\Sigma}^d \times \mathbb{C} : f_1(x) = \dots = f_{mn}(x) = 0\}.$$

Projection on the first factor gives the surjective maps  $\pi_n^{d,m} : Z_n^{d,m} \rightarrow \Sigma_n^{d,m}$  and  $\tilde{\pi}^d : \tilde{Z}^d \rightarrow \tilde{\Sigma}^d$ , respectively.

**Definition 3.2.** (i) Let  $\varphi : \mathbb{P}^d(m, n) \xrightarrow{\cong} \mathbb{C}^{dmn}$  be any fixed homeomorphism and define the embedding  $j_d : \tilde{Z}^d \rightarrow \mathbb{C}^{dmn+2d+1}$  by

$$(3.1) \quad j_d((f_1, \dots, f_{mn}), x) = (\varphi(f_1, \dots, f_{mn}), 1, x, x^2, \dots, x^{2d})$$

for  $((f_1, \dots, f_{mn}), x) \in \tilde{Z}^d$ . Similarly, define the embedding  $\tilde{i} : Z_n^{d,m} \rightarrow \mathbb{C}^{dmn+2d+1}$  by

$$(3.2) \quad \tilde{i}((f_1, \dots, f_m), x) = j((\mathbf{f}_1(z), \dots, \mathbf{f}_m(z)), x)$$

for  $((f_1, \dots, f_m), x) \in Z_n^{d,m}$ , where  $\mathbf{f}_k(z)$  denotes the  $n$ -tuple of polynomials defined in (1.7). Note that it is easy to see that the following holds:

$$(3.3) \quad j = \tilde{i} \circ (\hat{j}_n^{d,m} \times \text{id}_{\mathbb{C}})$$

where  $\hat{j}_n^{d,m} : \Sigma_n^{d,m} \rightarrow \tilde{\Sigma}^d$  denote the embedding defined by

$$(3.4) \quad \hat{j}_n^{d,m}(f_1(z), \dots, f_m(z)) = (\mathbf{f}_1(z), \dots, \mathbf{f}_m(z))$$

for  $(f_1(z), \dots, f_m(z)) \in \Sigma_n^{d,m}$ .

(ii) Let  $(\mathcal{X}^d, \pi^\Delta : \mathcal{X}^d \rightarrow \Sigma_n^{d,m})$  and  $(\tilde{\mathcal{X}}^d, \tilde{\pi}^\Delta : \tilde{\mathcal{X}}^d \rightarrow \tilde{\Sigma}^d)$  be the simplicial resolutions of  $(\pi_n^{d,m}, i)$  and  $(\tilde{\pi}, j)$ , respectively. Then it is easy to see that  $\mathcal{X}^d$  and  $\tilde{\mathcal{X}}^d$  are no-degenerate simplicial resolutions, and that there are two natural increasing filtrations

$$\begin{aligned}\emptyset &= \mathcal{X}_0^d \subset \mathcal{X}_1^d \subset \mathcal{X}_2^d \subset \dots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_k^d = \mathcal{X}^d, \\ \emptyset &= \tilde{\mathcal{X}}_0^d \subset \tilde{\mathcal{X}}_1^d \subset \tilde{\mathcal{X}}_2^d \subset \dots \subset \bigcup_{k=0}^{\infty} \tilde{\mathcal{X}}_k^d = \tilde{\mathcal{X}}^d,\end{aligned}$$

such that

$$(3.5) \quad \mathcal{X}_k^d = \mathcal{X}^d \quad \text{if } k > \left\lfloor \frac{d}{n} \right\rfloor \quad \text{and} \quad \tilde{\mathcal{X}}_k^d = \tilde{\mathcal{X}}^d \quad \text{if } k > d.$$

By Theorem 2.2, the map  $\pi_+^\Delta : \mathcal{X}_+^d \xrightarrow{\sim} \Sigma_{n+}^{d,m}$  is a homology equivalence. Since  $\mathcal{X}_{k+}^d / \mathcal{X}_{k-1+}^d \cong (\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d)_+$ , we have a spectral sequence

$$\{E_{t;d}^{k,s}, d_t : E_{t;d}^{k,s} \rightarrow E_{t;d}^{k+t,s+1-t}\} \Rightarrow H_c^{k+s}(\Sigma_n^{d,m}, \mathbb{Z}),$$

where  $E_{1;d}^{k,s} = \tilde{H}_c^{k+s}(\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d, \mathbb{Z})$  and  $H_c^k(X, \mathbb{Z})$  denotes the cohomology group with compact supports given by  $H_c^k(X, \mathbb{Z}) = H^k(X_+, \mathbb{Z})$ .

Since there is a homeomorphism  $\mathbb{P}^d(\mathbb{C})^m \cong \mathbb{C}^{dm}$ , by Alexander duality there is a natural isomorphism

$$(3.6) \quad \tilde{H}_k(\text{Poly}_n^{d,m}, \mathbb{Z}) \cong \tilde{H}_c^{2md-k-1}(\Sigma_n^{d,m}, \mathbb{Z}) \quad \text{for any } k.$$

By reindexing we obtain a spectral sequence

$$(3.7) \quad \{E_{k,s}^{t;d}, d^t : E_{k,s}^{t;d} \rightarrow E_{k+t,s+t-1}^{t;d}\} \Rightarrow H_{s-k}(\text{Poly}_n^{d,m}, \mathbb{Z}),$$

where  $E_{k,s}^{1;d} = \tilde{H}_c^{2md+k-s-1}(\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d, \mathbb{Z})$ .

By a complete similar method we also have the spectral sequence

$$(3.8) \quad \{\tilde{E}_{k,s}^{t;d}, \tilde{d}^t : \tilde{E}_{k,s}^t \rightarrow \tilde{E}_{k+t,s+t-1}^t\} \Rightarrow H_{s-k}(\text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}), \mathbb{Z}),$$

where  $\tilde{E}_{k,s}^1 = \tilde{H}_c^{2dmn+k-s-1}(\tilde{\mathcal{X}}_k^d \setminus \tilde{\mathcal{X}}_{k-1}^d, \mathbb{Z})$ .

For a space  $X$ , let  $F(X, k) \subset X^k$  denote the ordered configuration space given by

$$F(X, k) = \{(x_1, \dots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\}.$$

Let  $S_k$  be the symmetric group of  $k$  letters. Then the group  $S_k$  acts on  $F(X, k)$  by permuting coordinates and let  $C_k(X)$  denote the orbit space  $C_k(X) = F(X, k)/S_k$ . Let  $X^{\wedge k}$  denote the  $k$ -fold smash product of a space  $X$ , i.e.  $X^{\wedge k} = X \wedge \cdots \wedge X$  ( $k$ -times). Then  $S_k$  acts on  $X^{\wedge k}$  by the coordinate permutation and define  $D_k(X)$  by the equivariant half smash product

$$(3.9) \quad D_k(X) = F(\mathbb{C}, k)_+ \wedge_{S_k} X^{\wedge k}.$$

In particular, for  $X = S^1$  we set

$$(3.10) \quad D_k = D_k(S^1).$$

**Lemma 3.3.** *If  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$ ,  $\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d$  is homeomorphic to the total space of a real affine bundle  $\xi_{d,k}$  over  $C_k(\mathbb{C})$  with rank  $l_{d,k} = 2m(d - nk) + k - 1$ .*

*Proof.* The argument is exactly analogous to the one in the proof of [1, Lemma 4.4]. Namely, an element of  $\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d$  is represented by the  $(m+1)$ -tuple  $(f_1(z), \dots, f_m(z), u)$ , where  $(f_1(z), \dots, f_m(z))$  is an  $m$ -tuple of monic polynomials of the same degree  $d$  in  $\Sigma_n^{d,m}$  and  $u$  is an element of the interior of the span of the images of  $k$  distinct points  $\{x_1, \dots, x_k\} \in C_k(\mathbb{C})$  such that  $\{x_j\}_{j=1}^k$  are common roots of  $\{f_i(z)\}_{i=1}^m$  of multiplicity  $n$  under a suitable embedding. Since the  $k$  distinct points  $\{x_j\}_{j=1}^k$  are uniquely determined by  $u$ , by the definition of the non-degenerate simplicial resolution, there are projection maps  $\pi_{k,d} : \mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d \rightarrow C_k(\mathbb{C})$  defined by  $((f_1, \dots, f_m), u) \mapsto \{x_1, \dots, x_k\}$ .

Now suppose that  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$  and  $1 \leq i \leq m$ . Let  $c = \{x_j\}_{j=1}^k \in C_k(\mathbb{C})$  be any fixed element and consider the fibre  $\pi_{k,d}^{-1}(c)$ . If we consider the condition that a polynomial  $f_i(z) \in \mathbb{P}^d(\mathbb{C})$  is divided by the polynomial  $\prod_{j=1}^k (z - x_j)^n$ , then it is easy to see that this condition is equivalent to the following condition:

$$(3.11) \quad f_i^{(t)}(x_j) = 0 \quad \text{for } 0 \leq t < n, \ 1 \leq j \leq k.$$

In general, for each  $0 \leq t < n$  and  $1 \leq j \leq k$ , the condition  $f_i^{(t)}(x_j) = 0$  gives one linear condition on the coefficients of  $f_i(z)$ , and it determines an affine hyperplane in  $\mathbb{P}^d(\mathbb{C})$ . For example, if we set  $f_i(z) = z^d + \sum_{s=1}^d a_s z^{d-s}$ , then  $f_i(x_j) = 0$  for all  $1 \leq j \leq k$  if and only if

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{d-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & x_k & x_k^2 & x_k^3 & \cdots & x_k^{d-1} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} x_1^d \\ x_2^d \\ \vdots \\ x_k^d \end{bmatrix}$$

Similarly,  $f'_i(x_j) = 0$  for all  $1 \leq j \leq k$  if and only if

$$\begin{bmatrix} 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (d-1)x_1^{d-2} \\ 0 & 1 & 2x_2 & 3x_2^2 & \cdots & (d-1)x_2^{d-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 2x_k & 3x_k^2 & \cdots & (d-1)x_k^{d-2} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} dx_1^{d-1} \\ dx_2^{d-1} \\ \vdots \\ dx_k^{d-1} \end{bmatrix}$$

and  $f''_i(x_j) = 0$  for all  $1 \leq j \leq k$  if and only if

$$\begin{bmatrix} 0 & 0 & 2 & 6x_1 & \cdots & (d-1)(d-2)x_1^{d-3} \\ 0 & 0 & 2 & 6x_2 & \cdots & (d-1)(d-2)x_2^{d-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 2 & 6x_k & \cdots & (d-1)(d-2)x_k^{d-3} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} d(d-1)x_1^{d-2} \\ d(d-1)x_2^{d-2} \\ \vdots \\ d(d-1)x_k^{d-2} \end{bmatrix}$$

and so on. Since  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$  and  $\{x_j\}_{j=1}^k \in C_k(\mathbb{C})$ , it follows from the properties of Vandermonde matrices that the condition (3.11) gives exactly  $mnk$  affinely independent conditions on the coefficients of  $f_i(z)$ . Hence, we see that the space of  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in \mathbb{P}^d(\mathbb{C})^m$  of monic polynomials which satisfies the condition (3.11) for each  $1 \leq i \leq m$  is the intersection of  $mnk$  affine hyperplanes in general position, and it has codimension  $mnk$  in  $\mathbb{P}^d(\mathbb{C})^m$ . Therefore, the fibre  $\pi_{k,d}^{-1}(c)$  is homeomorphic to the product of an open  $(k-1)$ -simplex with the real affine space of dimension  $2m(d-nk)$ . Because we can also check that the local triviality holds, we see that  $\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d$  is a real affine bundle over  $C_k(\mathbb{C})$  of rank  $l_{d,k} = 2m(d-nk) + k - 1$ .  $\square$

By using a complete similar method of Lemma 3.3 we can also prove the following result.

**Lemma 3.4.** *If  $1 \leq k \leq d$ ,  $\tilde{\mathcal{X}}_k^d \setminus \tilde{\mathcal{X}}_{k-1}^d$  is homeomorphic to the total space of a real affine bundle  $\tilde{\xi}_{d,k}$  over  $C_k(\mathbb{C})$  with rank  $\tilde{l}_{d,k} = 2mn(d-k) + k - 1$ .  $\square$*

**Lemma 3.5.** *If  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$ , there is a natural isomorphism*

$$E_{k,s}^{1;d} \cong H_{s-2(mn-1)k}(C_k(\mathbb{C}), \pm\mathbb{Z}),$$

where the twisted coefficients system  $\pm\mathbb{Z}$  comes from the Thom isomorphism.<sup>5</sup>

*Proof.* Suppose that  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$ . By Lemma 3.3, there is a homeomorphism  $(\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d)_+ \cong T(\xi_{d,k})$ , where  $T(\xi_{d,k})$  denotes the Thom space of

<sup>5</sup>The twisted coefficients system  $\pm\mathbb{Z}$  on  $C_k(\mathbb{C})$  is induced by the sign representation of the symmetric group. (cf. [31, page 114 and 254]).

$\xi_{d,k}$ . Since  $(2md + k - s - 1) - l_{d,k} = 2mnk - s$ , by using the Thom isomorphism and the Poincaré duality, there is a natural isomorphism  $E_{k,s}^{1;d} \cong \tilde{H}^{2md+k-s-1}(T(\xi_{d,k}), \mathbb{Z}) \cong H_c^{2mnk-s}(C_k(\mathbb{C}), \pm\mathbb{Z}) \cong H_{s-2(mn-1)k}(C_k(\mathbb{C}), \pm\mathbb{Z})$  and this completes the proof.  $\square$

A similar method also proves the following:

**Lemma 3.6.** *If  $1 \leq k \leq d$ , there is a natural isomorphism*

$$\tilde{E}_{k,s}^1 \cong H_{s-2(N-1)k}(C_k(\mathbb{C}), \pm\mathbb{Z}). \quad \square$$

**Corollary 3.7.** (i) *There is a natural isomorphism*

$$E_{k,s}^{1;d} \cong \begin{cases} \mathbb{Z} & \text{if } (k, s) = (0, 0) \\ H_{s-2(mn-1)k}(C_k(\mathbb{C}), \pm\mathbb{Z}) & \text{if } 1 \leq k \leq \lfloor \frac{d}{n} \rfloor, s \geq 2(mn-1)k \\ 0 & \text{otherwise} \end{cases}$$

(ii) *There is a natural isomorphism*

$$\tilde{E}_{k,s}^1 \cong \begin{cases} \mathbb{Z} & \text{if } (k, s) = (0, 0) \\ H_{s-2(mn-1)k}(C_k(\mathbb{C}), \pm\mathbb{Z}) & \text{if } 1 \leq k \leq d, s \geq 2(mn-1)k \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* It is easy to see that the assertion (i) follows from Lemma 3.5 and (3.5). The assertion (ii) follows similarly.  $\square$

**Remark 3.8.** By using the complete similar way, for an integer  $N \geq 2$  one can obtain the spectral sequence

$$\{\tilde{E}_{k,s}^{t;N}, d^t : \tilde{E}_{k,s}^{t;N} \rightarrow \tilde{E}_{k+t,s+t-1}^{t;N}\} \Rightarrow H_{s-k}(\text{Hol}_d^*(S^2, \mathbb{CP}^{N-1}), \mathbb{Z})$$

such that

$$(3.12) \quad \tilde{E}_{k,s}^{1;N} \cong \begin{cases} \mathbb{Z} & \text{if } (k, s) = (0, 0) \\ H_{s-2(N-1)k}(C_k(\mathbb{C}), \pm\mathbb{Z}) & \text{if } 1 \leq k \leq d, s \geq 2(N-1)k \\ 0 & \text{otherwise} \end{cases} \quad \square$$

**Remark 3.9.** One can show that for  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$  there is a homeomorphism

$$(3.13) \quad (\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d) \times \mathbb{C}^{dm(n-1)} \cong \tilde{\mathcal{X}}_k^d \setminus \tilde{\mathcal{X}}_{k-1}^d.$$

Hence, it is easy to see that there is an isomorphism  $E_{k,s}^{1;d} \cong \tilde{E}_{k,s}^1$  for any  $s$  if  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$  (but this is easily seen by Lemma 3.5 and Lemma 3.6).  $\square$

## 4 Stabilization maps

**Definition 4.1.** Let

$$(4.1) \quad s_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Poly}_n^{d+1,m}$$

denote the stabilization map given by adding the points from the infinity as in [29, §5 page 57]. Similarly, let

$$(4.2) \quad s_d : \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}) \rightarrow \text{Hol}_{d+1}^*(S^2, \mathbb{CP}^{mn-1})$$

be the stabilization map given in [29]. □

It is easy to see that the following diagram is commutative (up to homotopy)

$$(4.3) \quad \begin{array}{ccc} \text{Poly}_n^{d,m} & \xrightarrow{s_n^{d,m}} & \text{Poly}_n^{d+1,m} \\ j_n^{d,m} \downarrow & & j_n^{d+1,m} \downarrow \\ \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}) & \xrightarrow{s_d} & \text{Hol}_{d+1}^*(S^2, \mathbb{CP}^{mn-1}) \end{array}$$

Note that the map  $s_n^{d,m}$  clearly extends to the map  $\text{P}^d(\mathbb{C})^m \rightarrow \text{P}^{d+1}(\mathbb{C})^m$  and its restriction gives the stabilization map  $\tilde{s}_n^{d,m} : \Sigma_n^{d,m} \rightarrow \Sigma_n^{d+1,m}$  between discriminants. It is easy to see that it also extends to the open embedding

$$(4.4) \quad \tilde{s}_n^{d,m} : \Sigma_n^{d,m} \times \mathbb{C}^m \rightarrow \Sigma_n^{d+1,m}.$$

Since the one-point compactification is contravariant for open embeddings, it induces the map

$$(4.5) \quad \tilde{s}_{n+}^{d,m} : (\Sigma_n^{d+1,m})_+ \rightarrow (\Sigma_n^{d,m} \times \mathbb{C}^m)_+ = \Sigma_{n+}^{d,m} \wedge S^{2m}$$

between one-point compactifications. Then we obtain the following diagram is commutative

$$(4.6) \quad \begin{array}{ccc} \tilde{H}_k(\text{Poly}_n^{d,m}, \mathbb{Z}) & \xrightarrow{s_{n+}^{d,m}} & \tilde{H}_k(\text{Poly}_n^{d+1,m}, \mathbb{Z}) \\ Al \downarrow \cong & & Al \downarrow \cong \\ \tilde{H}_c^{2dm-k-1}(\Sigma_n^{d,m}, \mathbb{Z}) & \xrightarrow{\tilde{s}_{n+}^{d,m*}} & \tilde{H}_c^{2(d+1)m-k-1}(\Sigma_n^{d+1,m}, \mathbb{Z}) \end{array}$$

where  $Al$  denotes the Alexander duality isomorphism and  $\tilde{s}_{n+}^{d,m*}$  denotes the composite of the the suspension isomorphism with the homomorphism  $(\tilde{s}_n^{d,m})^*$ ,

$$\tilde{H}_c^{2dm-k-1}(\Sigma_n^{d,m}) \xrightarrow{\cong} \tilde{H}_c^{2(d+1)m-k-1}(\Sigma_n^{d,m} \times \mathbb{C}^m) \xrightarrow{(\tilde{s}_{n+}^{d,m})^*} \tilde{H}_c^{2(d+1)m-k-1}(\Sigma_n^{d+1,m}).$$

Note that the map  $\hat{s}_n^{d,m}$  induces the filtration preserving map

$$(4.7) \quad \hat{s}_n^{d,m} : \mathcal{X}^d \times \mathbb{C}^m \rightarrow \mathcal{X}^{d+1}$$

and it defines the homomorphism of spectral sequences

$$(4.8) \quad \{\theta_{k,s}^t : E_{k,s}^{t;d} \rightarrow E_{k,s}^{t;d+1}\}.$$

**Lemma 4.2.** *If  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$ ,  $\theta_{k,s}^1 : E_{k,s}^{1;d} \xrightarrow{\cong} E_{k,s}^{1;d+1}$  is an isomorphism for any  $s$ .*

*Proof.* Suppose that  $1 \leq k \leq \lfloor \frac{d}{n} \rfloor$ . If we set  $\hat{s}_{n,k}^{d,m} = \hat{s}_n^{d,m}|_{\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d}$ , the diagram

$$\begin{array}{ccc} (\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d) \times \mathbb{C}^m & \xrightarrow{\pi_{k,d}} & C_k(\mathbb{C}) \\ \hat{s}_{n,k}^{d,m} \downarrow & & \parallel \\ \mathcal{X}_k^{d+1} \setminus \mathcal{X}_{k-1}^{d+1} & \xrightarrow{\pi_{k,d+1}} & C_k(\mathbb{C}) \end{array}$$

is commutative. Hence,  $\theta_{k,s}^1$  is an isomorphism.  $\square$

Now we prove the main key result.

**Theorem 4.3.** *If  $n \geq 2$ , the stabilization map*

$$s_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Poly}_n^{d+1,m}$$

*is a homology equivalence for  $\lfloor \frac{d}{n} \rfloor = \lfloor \frac{d+1}{n} \rfloor$ , and it is a homology equivalence through dimension  $D(d; m, n)$  for  $\lfloor \frac{d}{n} \rfloor < \lfloor \frac{d+1}{n} \rfloor$ .*

*Proof.* First, consider the case  $\lfloor \frac{d}{n} \rfloor = \lfloor \frac{d+1}{n} \rfloor$ . In this case, by using Corollary 3.7 and Lemma 4.2 it is easy to see that  $\theta_{k,s}^1 : E_{k,s}^{1;d} \xrightarrow{\cong} E_{k,s}^{1;d+1}$  is an isomorphism for any  $(k, s)$ . Hence,  $\theta_{k,s}^\infty$  is an isomorphism for any  $(k, s)$ . Since  $\theta_{k,s}^t$  is induced from  $\hat{s}_n^{d,m}$ , it follows from (4.6) that the map  $s_n^{d,m}$  is a homology equivalence.

Next assume that  $\lfloor \frac{d}{n} \rfloor < \lfloor \frac{d+1}{n} \rfloor$ , i.e.  $\lfloor \frac{d+1}{n} \rfloor = \lfloor \frac{d}{n} \rfloor + 1$ . In this case, by considering the differential  $d^t : E_{k,s}^{t;d+\epsilon} \rightarrow E_{k+t,s+t-1}^{t;d+\epsilon}$  ( $\epsilon \in \{0, 1\}$ ), Lemma 4.2 and Corollary 3.7, we easily see that  $\theta_{k,s}^t : E_{k,s}^{t;d} \rightarrow E_{k,s}^{t;d+1}$  is an isomorphism for any  $(k, s)$  and  $t$  as long as the condition  $s - t \leq D(d; m, n)$  is satisfied. Hence, if  $s - t \leq D(d; m, n)$ ,  $\theta_{k,s}^\infty$  is always an isomorphism and so that the map  $s_n^{d,m}$  is a homology equivalence through dimension  $D(d; m, n)$ .  $\square$

**Theorem 4.4** ([21], Theorem 2.8). *If  $n \geq 2$ , the stabilization map*

$$s_d : \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}) \rightarrow \text{Hol}_{d+1}^*(S^2, \mathbb{CP}^{mn-1})$$

*is a homology equivalence through dimension  $(2mn - 3)(d + 1) - 1$ .*  $\square$



**Definition 4.5.** Let  $\text{Poly}_n^{\infty, m}$  denote the colimit  $\lim_{d \rightarrow \infty} \text{Poly}_n^{d, m}$  taken from the stabilization maps  $s_n^{d, m}$ 's. Then the natural map  $i_n^{d, m} : \text{Poly}_n^{d, m} \rightarrow \Omega_d^2 \mathbb{CP}^{mn-1}$  (given in (1.8)) induces the map

$$(4.9) \quad i_n^{\infty, m} = \varinjlim i_n^{d, m} : \text{Poly}_n^{\infty, m} \rightarrow \lim_{d \rightarrow \infty} \Omega_d^2 \mathbb{CP}^{mn-1} \simeq \Omega^2 S^{2mn-1}.$$

Then we have the following result whose proof is given in the next section.

**Theorem 4.6.** *If  $n \geq 2$ , the map  $i_n^{\infty, m} : \text{Poly}_n^{\infty, m} \xrightarrow{\simeq} \Omega^2 S^{2mn-1}$  is a homology equivalence.*

## 5 Scanning maps and the unstable results

In this section, we prove Theorem 4.6 by using the scanning maps. Next we give the proofs of the stability results (Theorem 1.8 and Corollary 1.9).

**Definition 5.1.** For a space  $X$  let  $\text{SP}^d(X)$  denote the  $d$ -th symmetric product defined by the quotient space  $\text{SP}^d(X) = X^d/S_d$ , where the symmetric group  $S_d$  of  $d$ -letters acts on  $X^d$  by the permutation of coordinates. Note that there is a natural inclusion  $C_d(X) \subset \text{SP}^d(X)$ .

**Remark 5.2.** (i) An element of  $\text{SP}^d(X)$  may be identified with the formal linear combination  $\alpha = \sum_{i=1}^k d_k x_i$  ( $\{x_i\} \in C_k(X)$ ,  $\sum_{k=1}^l d_k = d$ ). We shall refer to  $\alpha$  as configuration of points, the point  $x_i$  having a multiplicity  $d_i$ .

(ii) If  $X = \mathbb{C}$ , then  $\text{P}^d(\mathbb{C})$  can be easily identified with the space  $\text{SP}^d(\mathbb{C})$  by the correspondence  $\prod_{i=1}^k (z - \alpha_i)^{d_i} \mapsto \sum_{i=1}^k d_i \alpha_i$ .  $\square$

**Definition 5.3.** For a space  $X$ , define the space  $\text{Pol}_n^{d, m}(X)$  by

$$(5.1) \quad \text{Pol}_n^{d, m}(X) = \{(\xi_1, \dots, \xi_m) \in \text{SP}^d(X)^m : (*)\},$$

where the condition  $(*)$  is given by

$$(*) \quad \cap_{i=1}^m \xi_i \text{ does not contain any element of multiplicity } \geq n.$$

**Remark 5.4.** By identifying  $\text{P}^d(\mathbb{C}) = \text{SP}^d(\mathbb{C})$  as in Remark 5.2, we easily see that there is a homeomorphism

$$(5.2) \quad \text{Poly}_n^{d, m} \cong \text{Pol}_n^{d, m}(\mathbb{C}).$$

**Definition 5.5.** If  $A \subset X$  is a closed subspace, we define

$$(5.3) \quad \text{Pol}_n^{d, m}(X, A) = \text{Pol}_n^{d, m}(X) / \sim,$$

where the equivalence relation “ $\sim$ ” is defined by

$$(\xi_1, \dots, \xi_m) \sim (\eta_1, \dots, \eta_m) \quad \text{if} \quad \xi_i \cap (X \setminus A) = \eta_i \cap (X \setminus A)$$

for each  $1 \leq i \leq m$ . Therefore, points in  $A$  are “ignored”. When  $A \neq \emptyset$ , there is a natural inclusion

$$\text{Pol}_n^{d,m}(X, A) \subset \text{Pol}_n^{d+1,m}(X, A)$$

by adding points in  $A$ . Define the space  $\text{Pol}_n^m(X, A)$  by the union

$$(5.4) \quad \text{Pol}_n^m(X, A) = \bigcup_{d \geq 1} \text{Pol}_n^{d,m}(X, A).$$

**Remark 5.6.** As a set,  $\text{Pol}_n^m(X, A)$  is bijectively equivalent to the disjoint union  $\bigcup_{d \geq 1} \text{Pol}_n^{d,m}(X \setminus A)$ . But these two spaces are not homeomorphic. For example, if  $X$  is connected, then  $\text{Pol}_n^m(X, A)$  is connected.  $\square$

We need two kinds of scanning maps. First, we define the scanning map for configuration space of particles.

**Definition 5.7.** Let us identify  $D^2 = \{x \in \mathbb{C} : |x| \leq 1\}$ , and let  $\epsilon > 0$  be a fixed sufficiently small positive number. Then for each  $w \in \mathbb{C}$ , let  $U_w$  be the open set  $U_w = \{x \in \mathbb{C} : |x - w| < \epsilon\}$ . Now define the scanning map

$$(5.5) \quad sc_n^{d,m} : \text{Pol}_n^{d,m}(\mathbb{C}) \rightarrow \Omega^2 \text{Pol}_n^m(D^2, S^1)$$

as follows. Let  $\alpha = (\xi_1, \dots, \xi_m) \in \text{Pol}_n^{d,m}(\mathbb{C})$ . Then let

$$sc_n^{d,m}(\alpha) : S^2 = \mathbb{C} \cup \infty \rightarrow \text{Pol}_n^m(D^2, S^1)$$

denote the map given by

$$w \mapsto (\xi_1 \cap \overline{U}_w, \dots, \xi_m \cap \overline{U}_w) \in \text{Pol}_n^m(\overline{U}_w, \partial \overline{U}_w) \cong \text{Pol}_n^m(D^2, S^1)$$

for  $w \in \mathbb{C}$ , where we use the canonical identification  $(\overline{U}_w, \partial \overline{U}_w) \cong (D^2, S^1)$ . Since  $\lim_{w \rightarrow \infty} sc_n^{d,m}(\alpha)(w) = (\emptyset, \dots, \emptyset)$ , we set  $sc_n^{d,m}(\alpha)(\infty) = (\emptyset, \dots, \emptyset)$  and we obtain the based map  $sc_n^{d,m}(\alpha) \in \Omega^2 \text{Pol}_n^m(D^2, S^1)$ .

Since the space  $\text{Pol}_n^{d,m}(\mathbb{C})$  is connected, the image of  $sc_n^{d,m}$  is contained in some path-component of  $\Omega^2 \text{Pol}_n^m(D^2, S^1)$ , which is denoted by  $\Omega_d^2 \text{Pol}_n^m(D^2, S^1)$ . Hence, finally we obtain the map

$$(5.6) \quad sc_n^{d,m} : \text{Pol}_n^{d,m}(\mathbb{C}) \rightarrow \Omega_d^2 \text{Pol}_n^m(D^2, S^1).$$

Now we identify  $\text{Poly}_n^{d,m} \cong \text{Pol}_n^{d,m}(\mathbb{C})$  as in (5.2) and by setting  $S = \lim_{d \rightarrow \infty} sc_n^{d,m}$ , we obtain *the scanning map*

$$(5.7) \quad S : \text{Poly}_n^{\infty,m} \rightarrow \lim_{d \rightarrow \infty} \Omega_d^2 \text{Pol}_n^m(D^2, S^1) \simeq \Omega_0^2 \text{Pol}_n^m(D^2, S^1).$$

**Theorem 5.8** ([16], [29]). *If  $n \geq 2$ , the scanning map*

$$S = \lim_{d \rightarrow \infty} sc_n^{d,m} : \text{Poly}_n^{\infty,m} \xrightarrow{\cong} \Omega_0^2 \text{Pol}_n^m(D^2, S^1)$$

*is a homology equivalence.*

*Proof.* The proof is similar to the argument of [29, §3]. Alternatively, we can prove this by using the complete similar way as in [16, page 99-100]  $\square$

**Definition 5.9.** (i) Let  $\mathcal{P}^d(\mathbb{C})$  denote the space of (not necessarily monic) all polynomials  $f(z) = \sum_{i=0}^d a_i z^i \in \mathbb{C}[z]$  of degree exactly  $d$  and let  $\text{Poly}_n^{d,m}$  denote the space of all  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in \mathcal{P}^d(\mathbb{C})^m$  such that polynomials  $\{f_1(z), \dots, f_m(z)\}$  have no common root of multiplicity  $\geq n$ .

(ii) For each nonempty open set  $X \subset \mathbb{C}$ , let  $\text{Poly}_n^m(X)$  denote the space of all  $m$ -tuples  $(f_1(z), \dots, f_m(z))$  satisfying the following two conditions:

(5.9.1)  $f_i(z) \in \mathbb{C}[z]$  is a complex polynomial of the same degree and it is not identically zero for each  $1 \leq i \leq m$ .

(5.9.2) Polynomials  $\{f_1(z), \dots, f_m(z)\}$  have no common root in  $X$  of multiplicity  $\geq n$ .

When  $X = \mathbb{C}$ , we write  $\text{Poly}_n^{d,m} = \text{Poly}_n^{d,m}(\mathbb{C})$ .  $\square$

**Remark 5.10.** (i) Note that  $\text{Poly}_n^m(\mathbb{C})$  is bijectively equivalent to the union  $\bigcup_{d \geq 0} \text{Poly}_n^{d,m}(\mathbb{C})$ , but these spaces are not homeomorphic because  $\text{Poly}_n^m(\mathbb{C})$  is connected.

(ii) It is easy to see that there are homeomorphisms

$$\mathcal{P}^d(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{P}^d(\mathbb{C}) \quad \text{and} \quad \text{Poly}_n^{d,m}(\mathbb{C}) \cong \mathbb{T}^m \times \text{Poly}_n^{d,m},$$

where we set  $\mathbb{T}^m = (\mathbb{C}^*)^m$ .  $\square$

Next consider the scanning map for algebraic maps.

**Definition 5.11.** (i) Let  $U = D^2 \setminus S^1 = \{x \in \mathbb{C} : |x| < 1\}$  and define *the scanning map*

$$(5.8) \quad sc_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Map}(\mathbb{C}, \text{Poly}_n^{d,m}(U))$$

for  $\text{Poly}_n^{d,m}$  by

$$sc_n^{d,m}(f_1(z), \dots, f_m(z))(w) = (f_1|_{U_w}, \dots, f_m|_{U_w})$$

for  $((f_1(z), \dots, f_m(z)), w) \in \text{Poly}_n^{d,m} \times \mathbb{C}$ , where we also use the canonical identification  $U \cong U_w$  as in the definition of the earlier scanning map.

(ii) Let  $q : \text{Poly}_n^m(\mathbb{C}) \rightarrow \text{Pol}_n^m(D^2, S^1)$  denote the map given by assigning to an  $m$ -tuples of polynomials their corresponding roots in  $U$ .  $\square$

**Lemma 5.12.** *The map  $q : \mathcal{Poly}_n^m(\mathbb{C}) \rightarrow \text{Pol}_n^m(D^2, S^1)$  is a quasifibration with fibre  $\mathbb{T}^m$ .*

*Proof.* This may be proved by using the well-known criterion of Dold-Thom as in the proof of [29, Lemma 3.3]. In fact, we can prove this by using the induction on the number  $m$ . The case  $m = 1$  is proved in [16, page 101]. Now assume that the case  $m - 1$  is true and filter the base space  $\text{Pol}_n^{d,m}(D^2, S^1)$  by the points of  $U$  in the first coordinate. Note that  $q$  is a trivial fibration over each the successive difference of the filtration. Then the Dold-Thom "attaching map" has the effect of multiplying polynomials with no root in  $U$  by a fixed polynomial  $z - \alpha$ , where  $\alpha \in \mathbb{C} \setminus U$ . Since  $\alpha$  may be moved continuously to 1, we can show the assertion in the same way as the case  $m = 1$ .  $\square$

**Definition 5.13.** (i) Let  $ev_0 : \mathcal{Poly}_n^m(U) \rightarrow \mathbb{C}^{mn} \setminus \{\mathbf{0}\}$  denote the evaluation map at  $z = 0$  given by

$$ev_0(f_1(z), \dots, f_m(z)) = (\mathbf{f}_1(0), \mathbf{f}_2(0) \cdots, \mathbf{f}_m(0))$$

for  $(f_1(z), \dots, f_m(z)) \in \mathcal{Poly}_n^{d,m}(U)$ .

(ii) Let  $G$  be a group and  $X$  a  $G$ -space. Then we denote by  $X//G$  the homotopy quotient of  $X$  by  $G$ ,  $X//G = EG \times_G X$ .  $\square$

**Remark 5.14.** Let  $\mathbb{T}^m = (\mathbb{C}^*)^m$  and consider the natural  $\mathbb{T}^m$ -actions on the spaces  $\mathcal{Poly}_n^m(U)$  and  $\mathbb{C}^{mn} \setminus \{\mathbf{0}\}$  given by the usual coordinate-wise multiplications. Then it is easy to see that  $ev_0$  is a  $\mathbb{T}^m$ -equivariant map.

**Lemma 5.15.** *The map  $ev_0 : \mathcal{Poly}_n^m(U) \rightarrow \mathbb{C}^{mn} \setminus \{\mathbf{0}\}$  is a homotopy equivalence.*

*Proof.* If  $m = 1$ , this is proved in [16, Theorem 2.4 (page 102-103)]. By using the same method with [18, Prop. 1], we can show the assertion for  $m \geq 2$ .  $\square$

Now we can prove Theorem 4.6.

*Proof of Theorem 4.6.* Note that  $\mathbb{T}^m$  does not act on  $(\mathbb{C}^*)^{mn} \setminus \{\mathbf{0}\}$  freely. Hence, by using its homotopy quotient, we have the commutative diagram

$$\begin{array}{ccccc}
\mathcal{Poly}_n^{d,m} & \xrightarrow{sc_n^{d,m}} & \text{Map}(\mathbb{C}, \mathcal{Poly}_n^m(U)) & \xrightarrow[\simeq]{ev_0} & \text{Map}(\mathbb{C}, \mathbb{C}^{mn} \setminus \{\mathbf{0}\}) \\
q_1 \downarrow & & q_2 \downarrow & & q_3 \downarrow \\
\mathcal{Poly}_n^{d,m}/\mathbb{T}^m & \longrightarrow & \text{Map}(\mathbb{C}, \mathcal{Poly}_n^m(U)/\mathbb{T}^m) & \xrightarrow[\simeq]{\tilde{ev}_0} & \text{Map}(\mathbb{C}, (\mathbb{C}^{mn} \setminus \{\mathbf{0}\})//\mathbb{T}^m) \\
\cong \downarrow & & q' \downarrow \simeq & & \\
\text{Poly}_n^{d,m} & \xrightarrow{sc_n^{d,m}} & \text{Map}(\mathbb{C}, \text{Pol}_n^m(D^2, S^1)) & & 
\end{array}$$

where the vertical maps  $q_i$  ( $i = 1, 2, 3$ ) are induced maps from the corresponding group actions, and  $q'$  is induced map from the map  $q$ .

Note that the map  $q'$  is a homotopy equivalence by Lemma 5.12. Since the map  $ev_0$  is  $\mathbb{T}^m$ -equivariant and a homotopy equivalence by Lemma 5.15, the map  $\tilde{ev}_0$  is a homotopy equivalence.

Now consider the map  $\gamma$  given by the second row of the above diagram after imposing the base point condition at  $\infty$ . If  $d \rightarrow \infty$ ,  $sc_n^{d,m}$  is a homology equivalence by Theorem 5.8. So the map  $\gamma$  is a homotopy equivalence if  $d \rightarrow \infty$ . However, since the map  $q_3$  induces a homotopy equivalence  $\Omega^2 S^{2mn-1} \simeq \Omega_0^2((\mathbb{C}^{mn} \setminus \{0\})/\mathbb{T}^m)$  (after imposing the base point condition at  $\infty$ ), this map coincides the map  $i_n^{\infty,m}$  (if  $d \rightarrow \infty$ ) up to homotopy equivalence. Hence,  $i_n^{\infty,m}$  is a homology equivalence.  $\square$

Next, we shall prove the unstable results (Theorem 1.8 and Corollary 1.9). For this purpose, remark the following two results.

**Lemma 5.16.** *If  $m \geq 2$  and  $n \geq 2$ , the space  $\text{Poly}_n^{d,m}$  is simply connected.*

*Proof.* Assume that  $m \geq 2$  and  $n \geq 2$ . Then by using the braid representation as in [15, §5. Appendix], any different kinds of strings can pass through and one can show that  $\pi_1(\text{Poly}_n^{d,m})$  is an abelian group. Hence, there is an isomorphism  $\pi_1(\text{Poly}_n^{d,m}) \cong H_1(\text{Poly}_n^{d,m}, \mathbb{Z})$ . Now consider the spectral sequence (3.7). Then it follows from Corollary 3.7 that  $H_k(\text{Poly}_n^{d,m}, \mathbb{Z}) = 0$  for any  $1 \leq k \leq 2mn - 5$ . Thus, the space  $\text{Poly}_n^{d,m}$  is simply connected.  $\square$

**Theorem 5.17.** *If  $m \geq 2$  and  $n \geq 2$ , the stabilization map*

$$s_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \text{Poly}_n^{d+1,m}$$

*is a homotopy equivalence if  $\lfloor \frac{d}{n} \rfloor = \lfloor \frac{d+1}{n} \rfloor$  and a homotopy equivalence through dimension  $D(d; m, n)$  otherwise.*

*Proof.* This follows from Theorem 4.3 and Lemma 5.16.  $\square$

Now we can complete the proof of Theorem 1.8 and Corollary 1.9.

*Proof of Theorem 1.8.* Since the case  $(m, n) = (3, 1)$  and the case  $(m, n) = (1, 3)$  were already proved in Theorem 1.4 and Theorem 1.5, without loss of generalities, we may assume that  $m \geq 2$  and  $n \geq 2$ . It follows from Theorem 4.3 and Theorem 4.6 that the map  $i_n^{d,m} : \text{Poly}_n^{d,m} \rightarrow \Omega_0^2 \mathbb{CP}^{mn-1}$  is a homology equivalence through dimension  $D(d; m, n)$ . However, since  $\text{Poly}_n^{d,m}$  and  $\Omega_0^2 \mathbb{CP}^{mn-1} \simeq \Omega^2 S^{2mn-1}$  are simply connected, the map  $i_n^{d,m}$  is indeed a homotopy equivalence through dimension  $D(d; m, n)$ .  $\square$

*Proof of Corollary 1.9.* By the same reason as the proof of Theorem 1.8, it suffices to prove the assertion when  $m \geq 2$  and  $n \geq 2$ . By the diagram (4.3), it is easy to see that the following diagram is commutative (up to homotopy).

$$\begin{array}{ccc} \text{Poly}_n^{d,m} & \xrightarrow{i_n^{d,m}} & \Omega_d^2 \mathbb{CP}^{mn-1} \\ j_n^{d,m} \downarrow & & \parallel \\ \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}) & \xrightarrow{i'_d} & \Omega_d^2 \mathbb{CP}^{mn-1} \end{array}$$

It follows from Theorem 1.5 that  $i'_d$  is a homotopy equivalence through dimension  $(2mn-3)(d+1)-1$ . Moreover, by Theorem 1.8 we know that  $i_n^{d,m}$  is a homotopy equivalence through dimension  $D(d; m, n)$ . Since  $D(d; m, n) < (2mn-3)(d+1)-1$ , it follows from above diagram that the map  $j_n^{d,m}$  is a homotopy equivalence through dimension  $D(d; m, n)$ .  $\square$

## 6 $C_2$ -structures

In this section we shall prove Theorem 1.10 and Theorem 1.11.

**Definition 6.1.** Let  $J = (0, 1)$  be an open unit disk in  $\mathbb{R}$ . Recall that a *little 2-cube*  $c$  means an affine embedding  $c : J^2 \rightarrow J^2$  with parallel axes.

(i) For each integer  $j \geq 1$ , let  $C_2(j)$  denote the space consisting of all  $j$ -tuples  $(c_1, \dots, c_j)$  of little 2-cubes such that  $c_k(J^2) \cap c_i(J^2) = \emptyset$  if  $k \neq i$ .

(ii) Let  $\alpha_0 : \mathbb{C} \xrightarrow{\cong} J^2$ . If  $S_j$  denotes the symmetric group of  $j$ -letters, it is easy to see that it acts on the spaces  $C_2(j)$  and  $\text{Poly}_n^{d,m}$  by the permutation of coordinates in a natural way. Then we identify  $\mathbb{C} = \mathbb{R}^2$  and define the *structure map*  $\mathcal{I}_j : C_2(j) \times_{S_j} (\text{Poly}_n^{d,m})^j \rightarrow \text{Poly}_n^{dj,m}$  by

$$(6.1) \quad \mathcal{I}_j((c_1, \dots, c_j), (f_1, \dots, f_j)) = \left( \prod_{k=1}^j c_k(f_{1;k}(z)), \dots, \prod_{k=1}^j c_k(f_{m;k}(z)) \right)$$

for  $(c_1, \dots, c_j) \in C_2(j)$  and  $f_k = (f_{i;k}(z), \dots, f_{m;k}(z)) \in \text{Poly}_n^{d,m}$  ( $1 \leq k \leq j$ ), where we set

$$(6.2) \quad c_k(f(z)) = \prod_{i=1}^d (z - c_k \circ \alpha_0(a_i)) \quad \text{if } f(z) = \prod_{i=1}^d (z - a_i) \in \mathbb{P}^d(\mathbb{C}).$$

(iii) Similarly, let  $c_* = (c_1, c_2) \in C_2(2)$  be any fixed element and define the *loop product*  $\mu_{d_1, d_2} : \text{Poly}_n^{d_1, m} \times \text{Poly}_n^{d_2, m} \rightarrow \text{Poly}_n^{d_1+d_2, m}$  by

$$(6.3) \quad \mu_{d_1, d_2}(f, g) = (c_1(f_1(z))c_2(g_1(z)), \dots, c_1(f_m(z))c_2(g_m(z)))$$

for  $(f, g) = ((f_1(z), \dots, f_m(z)), (g_1(z), \dots, g_m(z))) \in \text{Poly}_n^{d_1, m} \times \text{Poly}_n^{d_2, m}$ .

**Remark 6.2.** (i) It is easy to see that  $\mu_{d_1, d_2}(f, g) = \mathcal{I}_2(c_*; f, g)$  if  $d_1 = d_2 = d$ .  
(ii) Let  $\text{Poly}_n^{0, m} = \{*_0\}$  and let  $\text{Poly}_n^m$  denote the disjoint union

$$(6.4) \quad \text{Poly}_n^m = \coprod_{d \geq 0} \text{Poly}_n^{d, m}.$$

If we set  $\mu_{d, 0}(f, *_0) = \mu_{0, d}(*_0, f) = f$  for any  $f \in \text{Poly}_n^{d, m}$ , it is easy to see that  $\text{Poly}_n^m$  is a homotopy associative H-space with unit  $*_0$ , and we can easily see that  $\{\text{Poly}_n^m, \mathcal{I}_j\}_{j \geq 1}$  is a  $C_2$ -operad space. Thus, by using the group completion Theorem and Theorem 1.8, we see that there is a homotopy equivalence

$$(6.5) \quad \Omega B(\text{Poly}_n^m) \simeq \Omega^2 \mathbb{CP}^{mn-1} \simeq \mathbb{Z} \times \Omega^2 S^{2mn-1}. \quad \square$$

**Definition 6.3.** Let

$$\begin{aligned} * : \quad & \text{Hol}_{d_1}^*(S^2, \mathbb{CP}^{mn-1}) \times \text{Hol}_{d_2}^*(S^2, \mathbb{CP}^{mn-1}) \rightarrow \text{Hol}_{d_1+d_2}^*(S^2, \mathbb{CP}^{mn-1}) \\ & \text{and} \\ \mathcal{I} : \quad & C_2(j) \times_{S_j} \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})^j \rightarrow \text{Hol}_{dj}^*(S^2, \mathbb{CP}^{mn-1}) \end{aligned}$$

denote the loop product and the  $C_2$ -structure map defined in [4, (4.8)] and [4, (4.10)], respectively.

It is easy to see that the above definitions of the loop products and the structure maps are completely analogous to those of [4, (4.8), (4.10)], and we have the following:

**Lemma 6.4.** (i) *The following two diagrams are homotopy commutative.*

$$\begin{array}{ccc} \text{Poly}_n^{d_1, m} \times \text{Poly}_n^{d_2, m} & \xrightarrow{\mu_{d_1, d_2}} & \text{Poly}_n^{d_1+d_2, m} \\ j_n^{d_1, m} \times j_n^{d_2, m} \downarrow & & j_n^{d_1+d_2, m} \downarrow \\ \text{Hol}_{d_1}^*(S^2, \mathbb{CP}^{mn-1}) \times \text{Hol}_{d_2}^*(S^2, \mathbb{CP}^{mn-1}) & \xrightarrow{*} & \text{Hol}_{d_1+d_2}^*(S^2, \mathbb{CP}^{mn-1}) \\ \\ C_2(j) \times_{S_j} (\text{Poly}_n^{d, m})^j & \xrightarrow{\mathcal{I}_j} & \text{Poly}_n^{dj, m} \\ 1 \times (j_n^{d, m})^j \downarrow & & j_n^{dj, m} \downarrow \\ C_j(2) \times_{S_j} \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})^j & \xrightarrow{\mathcal{I}} & \text{Hol}_{dj}^*(S^2, \mathbb{CP}^{mn-1}) \end{array}$$

(ii) *The map*

$$\coprod_{d \geq 0} i_n^{d, m} : \text{Poly}_n^m = \coprod_{d \geq 0} \text{Poly}_n^{d, m} \rightarrow \coprod_{d \in \mathbb{Z}} \Omega_d^2 \mathbb{CP}^{mn-1} = \Omega^2 \mathbb{CP}^{mn-1}$$

*is a  $C_2$ -map up to homotopy equivalence.*

*Proof.* This can be proved by an analogous way as given in [4, Theorem 4.10].  $\square$

**Lemma 6.5.** *There is a homotopy equivalence  $\text{Poly}_n^{n,m} \simeq S^{2mn-3}$ .*

*Proof.* This easily follows from the homeomorphism (1.9).  $\square$

**Lemma 6.6** ([7], [8], [30]). (i) *If  $X$  is a connected based CW complex,*

*there is a stable homotopy equivalence  $\Omega^2 \Sigma^2 X \simeq_s \bigvee_{k=1}^{\infty} D_k(X)$ , where  $D_k(X)$  denotes the space  $D_k(X) = F(\mathbb{C}, k)_+ \wedge_{S_k} (\bigwedge_{k=1}^k X)$  as in (3.9).*

(ii) *For integers  $k \geq 1$  and  $N \geq 2$ , there is a homotopy equivalence  $D_k(S^{2N-1}) \simeq \Sigma^{2(N-1)k} D_k$ , where  $D_k = D_k(S^1)$  as in (3.10).*

(iii) *The canonical projection  $p_{k,N} : F(\mathbb{C}, k) \times_{S_k} (S^{2N-1})^k \rightarrow D_k(S^{2N-1})$  has the stable section  $e_{k,N} : D_k(S^{2N-1}) \rightarrow F(\mathbb{C}, k) \times_{S_k} (S^{2N-1})^k$ .*  $\square$

**Definition 6.7.** (i) For each  $1 \leq k < d$ , let  $s_{k,d} : \text{Poly}_n^{kn,m} \rightarrow \text{Poly}_n^{dn,m}$  denote the composite of stabilization maps

$$(6.6) \quad s_{k,d} : \text{Poly}_n^{kn,m} \longrightarrow \text{Poly}_n^{kn+1,m} \longrightarrow \dots \longrightarrow \text{Poly}_n^{dn-1,m} \longrightarrow \text{Poly}_n^{dn,m}.$$

We denote by  $\text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}$  the mapping cone of the map  $s_{k,d}$ .

(ii) Let

$$(6.7) \quad e_d : \Sigma^{2(mn-2)d} D_d \rightarrow F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d$$

denote the stable map defined by the composite of maps

$$e_d : \Sigma^{2(mn-2)d} D_d \simeq D_d(S^{2mn-3}) \xrightarrow{e_{d,mn-1}} F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d.$$

It is easy to see that  $e_d$  is a stable section of the projection  $F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d \rightarrow D_d(S^{2mn-3}) \simeq \Sigma^{2(mn-2)d} D_d$ .

(iii) Since there is a  $S_d$ -equivariant homotopy equivalence  $C_2(d) \times_{S_d} F(\mathbb{C}, d)$ , one can define the stable map  $\Psi_d : \Sigma^{2(mn-2)d} D_d \rightarrow \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}$  by

$$(6.8) \quad \Psi_d = \tilde{p}_d \circ \mathcal{I}'_d \circ e_d,$$

where  $\tilde{p}_d : \text{Poly}_n^{dn,m} \rightarrow \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}$  is the natural projection and the map  $\mathcal{I}'_d$  denotes the map defined by the composite of maps

$$(6.9) \quad \mathcal{I}'_d : F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d \simeq C_2(d) \times_{S_d} (\text{Poly}_n^{n,m})^d \xrightarrow{\mathcal{I}_d} \text{Poly}_n^{dn,m}.$$



Note that the following diagram is commutative.

$$\begin{array}{ccc}
\Sigma^{2(mn-2)d} D_d \simeq_s D_d(S^{2mn-3}) & \xrightarrow{e_d} & F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d \\
\Psi_d \downarrow & & \mathcal{I}'_d \downarrow \\
\text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m} & \xleftarrow{\tilde{p}_d} & \text{Poly}_n^{dn,m}
\end{array}$$

Similarly, define the stable map  $\Phi_d : \bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k \rightarrow \text{Poly}_n^{dn,m}$  by

$$(6.10) \quad \Phi_d = (\vee s_{k,d}) \circ (\vee \mathcal{I}'_d) \circ (\vee e_k),$$

Thus the following diagram is commutative:

$$\begin{array}{ccccc}
\bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k & \xrightarrow[\simeq_s]{} & \bigvee_{k=1}^d D_k(S^{2mn-3}) & \xrightarrow{\vee e_k} & \bigvee_{k=1}^d F(\mathbb{C}, k) \times_{S_k} (S^{2mn-3})^k \\
\Phi_d \downarrow & & & & \vee \mathcal{I}'_k \downarrow \\
\text{Poly}_n^{dn,m} & \xrightarrow{=} & \text{Poly}_n^{dn,m} & \xleftarrow{\vee s_{k,d}} & \bigvee_{k=1}^d \text{Poly}_n^{kn,m}
\end{array}$$

(iv) For a connected space  $X$ , let  $J_2(X)$  denote *the May-Milgram model* for  $\Omega^2 \Sigma^2 X$  [25]

$$(6.11) \quad J_2(X) = \left( \prod_{k=1}^{\infty} F(\mathbb{C}, k) \times_{S_k} X^k \right) / \sim,$$

where  $\sim$  denotes the well-known equivalence relation. For each integer  $d \geq 1$ , let  $J_2(X)_d \subset J_2(X)$  denote *the  $d$ -th stage filtration of the May-Milgram model* for  $\Omega^2 \Sigma^2 X$  defined by

$$(6.12) \quad J_2(X)_d = \left( \prod_{k=1}^d F(\mathbb{C}, k) \times_{S_k} X^k \right) / \sim. \quad \square$$

**Theorem 6.8.** *The map  $\Psi_d : \Sigma^{2(mn-2)d} D_d \xrightarrow[\simeq_s]{} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}$  is a stable homotopy equivalence.*

The proof of Theorem 6.8 is postponed to §7, and we prove the following result.

**Theorem 6.9.** *The map  $\Phi_d : \bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k \xrightarrow[\simeq_s]{} \text{Poly}_n^{dn,m}$  is a stable homotopy equivalence.*

*Proof.* We proceed by the induction on  $d$ . If  $d = 1$ , since there is a homotopy equivalence  $D_1 \simeq S^1$ , the assertion follows from Lemma 6.5. Assume that the result holds for the case  $d - 1$ , i.e. the map  $\Phi_{d-1}$  is a stable homotopy equivalence. Note that the following diagram is commutative.

$$\begin{array}{ccc} \bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k & \xrightarrow{\Phi_d} & \text{Poly}_n^{dn,m} \\ \parallel & & \uparrow s_{d-1,d} \vee 1 \\ \left( \bigvee_{k=1}^{d-1} \Sigma^{2(mn-2)k} D_k \right) \vee \Sigma^{2(mn-2)d} D_d & \xrightarrow{\Phi_{d-1} \vee T'_d \circ e_d} & \text{Poly}_n^{(d-1)n,m} \vee \text{Poly}_n^{dn,m} \end{array}$$

Thus, we can easily obtain the following homotopy commutative diagram

$$\begin{array}{ccccc} \bigvee_{k=1}^{d-1} \Sigma^{2(mn-2)k} D_k & \xrightarrow{\subset} & \bigvee_{k=1}^{d-1} \Sigma^{2(mn-2)k} D_k & \longrightarrow & \Sigma^{2(mn-2)d} D_d \\ \Phi_{d-1} \downarrow \simeq_s & & \Phi_d \downarrow & & \Psi_d \downarrow \simeq_s \\ \text{Poly}_n^{(d-1)n,m} & \xrightarrow{s_{d-1,d}} & \text{Poly}_n^{dn,m} & \xrightarrow{\tilde{p}_d} & \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m} \end{array}$$

where the horizontal sequences are cofibration sequences. Since  $\Phi_{d-1}$  and  $\Psi_d$  are stable homotopy equivalences, the map  $\Phi_d$  is so.  $\square$

Now it is ready to prove Theorem 1.10.

*Proof of Theorem 1.10.* Since the case  $m = 1$  or the case  $n = 1$  were obtained by Theorem 1.6 and Theorem 1.7, assume that  $m \geq 2$  and  $n \geq 2$ . Then the assertion easily follows from Theorem 5.17 and Theorem 6.9.  $\square$

Next we shall prove Theorem 1.11.

**Definition 6.10.** It follows from Lemma 6.4 and [4, Theorem 4.14, Theorem 4.16] that  $C_2$ -structure of  $\text{Pol}_n^m = \coprod_{d \geq 0} \text{Poly}_n^{d,m}$  and that of  $J_2(S^{2mn-3})$  induced from the double loop product are compatible. So the structure maps  $\mathcal{I}_d$ 's induce a map

$$(6.13) \quad \epsilon_d : J_2(S^{2mn-3})_d \rightarrow \text{Poly}_n^{dn,m}$$

such that the following diagram is homotopy commutative:

$$(6.14) \quad \begin{array}{ccc} \bigvee_{k=1}^d F(\mathbb{C}, k) \times_{S_k} (S^{2mn-3})^k & \xrightarrow{\vee \mathcal{I}'_k} & \bigvee_{k=1}^d \text{Poly}_n^{kn,m} \\ \vee q_k \downarrow & & \vee s_{k,d} \downarrow \\ J_2(S^{2mn-3})_d & \xrightarrow{\epsilon_d} & \text{Poly}_n^{dn,m} \end{array}$$

where

$$(6.15) \quad q_k : F(\mathbb{C}, k) \times_{S_k} (S^{2mn-3})^k \rightarrow J_2(S^{2mn-3})_d \quad (1 \leq k \leq d)$$

denotes the natural projection.  $\square$

**Theorem 6.11.** *If  $m \geq 2$  and  $n \geq 2$ , the map  $\epsilon_d : J_2(S^{2mn-3})_d \xrightarrow{\simeq} \text{Poly}_n^{dn,m}$  is a homotopy equivalence.*

*Proof.* Since stable maps  $\{e_k\}$  are stable sections of the Snaith splittings (by Lemma 6.6), the map  $(\vee q_k) \circ (\vee e_k) : \bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k \xrightarrow{\simeq_s} J_2(S^{2mn-3})_d$  is a stable homotopy equivalence. Now, it is easy to see that the following diagram is stable homotopy commutative:

$$\begin{array}{ccccc}
\bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k & \xrightarrow{\vee e_k} & \bigvee_{k=1}^d F(\mathbb{C}, k) \times_{S_k} (S^{2mn-3})^k & \xrightarrow{\vee \mathcal{I}'_k} & \bigvee_{k=1}^d \text{Poly}_n^{kn,m} \\
\parallel & & \vee q_k \downarrow & & \vee s_{k,d} \downarrow \\
\bigvee_{k=1}^d \Sigma^{2(mn-2)k} D_k & \xrightarrow[\simeq_s]{(\vee q_k) \circ (\vee e_k)} & J_2(S^{2mn-3})_d & \xrightarrow{\epsilon_d} & \text{Poly}_n^{dn,m}
\end{array}$$

Since  $\Phi_d = (\vee s_{k,d}) \circ (\vee \mathcal{I}'_k) \circ (\vee e_k)$  and it is a stable homotopy equivalence (by Theorem 6.9), the map  $\epsilon_d$  is so. Thus, the map  $\epsilon_d$  is a homology equivalence. Since  $J_2(S^{2mn-3})_d$  and  $\text{Poly}_n^{dn,m}$  are simply connected (by Lemma 5.16), the map  $\epsilon_d$  is a homotopy equivalence.  $\square$

Now we can give the proof of Theorem 1.11.

*Proof of Theorem 1.11.* If  $(m, n) = (3, 1)$ , then  $\text{Poly}_n^{d,m} = \text{Hol}_d^*(S^2, \mathbb{CP}^2) = \text{Hol}_{[\frac{d}{n}]}^*(S^2, \mathbb{CP}^{mn-1})$  and the assertion holds. If  $(m, n) = (1, 3)$ , it was already proved by Theorem 1.7 and assume that  $m \geq 2$  and  $n \geq 2$ . It follows from Theorem 5.17 that it suffices to prove that there is a homotopy equivalence

$$(6.16) \quad \text{Poly}_n^{dn,m} \simeq \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1}).$$

By Theorem 6.11,  $\epsilon_d : J_2(S^{2mn-3})_d \xrightarrow{\simeq} \text{Poly}_n^{dn,m}$  is a homotopy equivalence. On the other hand, it follows from (ii) of Theorem 1.6 that there is a homotopy equivalence  $J_2(S^{2mn-3})_d \simeq \text{Hol}_d^*(S^2, \mathbb{CP}^{mn-1})$ . Hence, there is a homotopy equivalence (6.16) and the assertion is obtained.  $\square$

## 7 The space $\text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}$

In this section we give the proof of Theorem 6.8. Since the case  $m = 1$  and the case  $n = 1$  were already proved in [6, Theorem 1, Theorem 15] and [17, Theorem 2.9], in this section we always assume that  $m \geq 2$  and  $n \geq 2$ . Let

$$(7.1) \quad \iota_d : \text{Poly}_n^{dn,m} \rightarrow \text{Poly}_n^{\infty,m}$$

denote the natural inclusion map induced from the stabilization maps.

**Theorem 7.1.** *The stable map*

$$(\vee \iota_d) \circ (\vee \mathcal{I}'_d) \circ (\vee e_d) : \bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d \xrightarrow{\simeq_s} \text{Poly}_n^{\infty, m}$$

*is a stable homotopy equivalence.*

*Proof.* By using (ii) of Lemma 6.4 we see that  $i_n^{\infty, m} : \text{Poly}_n^{\infty, m} \rightarrow \Omega^2 S^{2mn-1}$  is a  $C_2$ -map such that the following diagram is homotopy commutative

$$\begin{array}{ccc} J_2(\text{Poly}_n^{\infty, m}) & \xrightarrow[\simeq]{J_2(i_n^{\infty, m})} & J_2(\Omega^2 S^{2mn-1}) \\ r_1 \downarrow & & r_2 \downarrow \\ \text{Poly}_n^{\infty, m} & \xrightarrow[\simeq]{i_n^{\infty, m}} & \Omega^2 S^{2mn-1} \end{array}$$

where  $r_1$  and  $r_2$  are natural retraction maps. Since  $m \geq 2$  and  $n \geq 2$ , the two maps  $i_n^{\infty, m}$  and  $J_2(i_n^{\infty, m})$  are indeed homotopy equivalences (by Theorem 4.6 and Lemma 5.16).

Similarly, by using (ii) of Lemma 6.4 we have the following homotopy commutative diagram

$$\begin{array}{ccccc} \bigvee_{d=1}^{\infty} F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d & \xrightarrow{\vee p_d} & J_2(S^{2mn-3}) & \xrightarrow{J_2(\iota)} & J_2(\text{Poly}_n^{\infty, m}) \\ \vee \mathcal{I}'_d \downarrow & & & & r_1 \downarrow \\ \bigvee_{d=1}^{\infty} \text{Poly}_n^{dn, m} & \xrightarrow{\vee \iota_d} & \text{Poly}_n^{\infty, m} & \xrightarrow{=} & \text{Poly}_n^{\infty, m} \end{array}$$

where  $\iota$  denotes the natural inclusion map  $\iota : S^{2mn-3} \simeq \text{Poly}_n^{n, m} \xrightarrow{\iota_1} \text{Poly}_n^{\infty, m}$ . Now consider the composite of maps

$$J_2(S^{2mn-3}) \xrightarrow{J_2(\iota)} J_2(\text{Poly}_n^{\infty, m}) \xrightarrow[\simeq]{J_2(i_n^{\infty, m})} J_2(\Omega^2 S^{2mn-1}) \xrightarrow{r_2} \Omega^2 S^{2mn-1}$$

Since the map  $i_n^{\infty, m} \circ \iota$  is the natural inclusion of the bottom cell of the double suspension  $E^2 : S^{2mn-3} \rightarrow \Omega^2 \Sigma^2 S^{2mn-3} = \Omega^2 S^{2mn-1}$  (up to homotopy equivalence), the map  $r_2 \circ J_2(i_n^{\infty, m}) \circ J_2(\iota)$  is homotopic to the natural homotopy equivalence  $J_2(S^{2mn-3}) \simeq \Omega^2 S^{2mn-1}$ . Thus the above two diagrams reduce to the following stable homotopy commutative diagram

$$\begin{array}{ccc} \bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d & & \\ \vee e_d \downarrow & & \\ \bigvee_{d=1}^{\infty} F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d & \xrightarrow{\vee p_d} & J_2(S^{2mn-3}) \simeq \Omega^2 S^{2mn-1} \\ \vee \mathcal{I}'_d \downarrow & & i_n^{\infty, m} \uparrow \simeq \\ \bigvee_{d=1}^{\infty} \text{Poly}_n^{dn, m} & \xrightarrow{\vee \iota_d} & \text{Poly}_n^{\infty, m} \end{array}$$

Since the stable maps  $\{e_d\}$  are stable sections of the stable homotopy equivalence  $\Omega^2 S^{2mn-1} \simeq_s \bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d$ , the map  $(\vee p_d) \circ (\vee e_d)$  is a stable homotopy equivalence and the map  $(\vee \iota_d) \circ (\vee \mathcal{I}'_d) \circ (\vee e_d)$  is so.  $\square$

**Remark 7.2.** Since  $\lim_{d \rightarrow \infty} \Phi_d = (\vee \iota_d) \circ (\vee \mathcal{I}'_d) \circ (\vee e_d)$ , the above result may be regarded as the stable version of Theorem 6.9.  $\square$

**Lemma 7.3.** (i) *The induced homomorphism*

$$(s_{d-1,d})_* : H_*(\text{Poly}_n^{(d-1)n,m}, \mathbb{Z}) \rightarrow H_*(\text{Poly}_n^{dn,m}, \mathbb{Z})$$

*is a monomorphism.*

(ii) *The induced homomorphism*

$$(\Psi_d)_* : H_*(\Sigma^{2(mn-2)d} D_d, \mathbb{F}) \rightarrow H_*(\text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{F})$$

*is a monomorphism for  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{Z}/p$  ( $p$  : any prime).*

We postpone the proof of Lemma 7.3 to §8 and we give the proof of Theorem 6.8.

*Proof of Theorem 6.8.* Let  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{Z}/p$  ( $p$  : any prime). By (i) of Lemma 7.3, there is an isomorphism of  $\mathbb{F}$ -vector spaces

$$H_*(\text{Poly}_n^{\infty,m}, \mathbb{F}) \cong H_*\left(\bigvee_{d=1}^{\infty} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{F}\right).$$

Hence, it follows from Lemma 6.6, Theorem 4.6 and Theorem 7.1 that there is an isomorphism of  $\mathbb{F}$ -vector spaces

$$H_*\left(\bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d, \mathbb{F}\right) \cong H_*\left(\bigvee_{d=1}^{\infty} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{F}\right).$$

Thus the following equality holds for each  $k \geq 1$ :

$$\dim_{\mathbb{F}} H_k\left(\bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d, \mathbb{F}\right) = \dim_{\mathbb{F}} H_k\left(\bigvee_{d=1}^{\infty} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{F}\right) < \infty$$

However, by (ii) of Lemma 7.3, we see that the homomorphism

$$(\vee_d \Psi_d)_* : H_*\left(\bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d, \mathbb{F}\right) \rightarrow H_*\left(\bigvee_{d=1}^{\infty} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{F}\right)$$

is injective. Therefore, the homomorphism

$$(\vee_d \Psi_d)_* : H_*\left(\bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d, \mathbb{F}\right) \xrightarrow{\cong} H_*\left(\bigvee_{d=1}^{\infty} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{F}\right)$$

is indeed an isomorphism for  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{Z}/p$  ( $p$  : any prime), and from the universal coefficient Theorem, the homomorphism

$$(\vee_d \Psi_d)_* : H_*\left(\bigvee_{d=1}^{\infty} \Sigma^{2(mn-2)d} D_d, \mathbb{Z}\right) \xrightarrow{\cong} H_*\left(\bigvee_{d=1}^{\infty} \text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{Z}\right)$$

is an isomorphism. Hence, for each  $d \geq 1$ , the homomorphism

$$(\Psi_d)_* : H_*(\Sigma^{2(mn-2)d} D_d, \mathbb{Z}) \xrightarrow{\cong} H_*(\text{Poly}_n^{dn,m} / \text{Poly}_n^{(d-1)n,m}, \mathbb{Z})$$

is an isomorphism. Therefore  $\Psi_d$  is a stable homotopy equivalence.  $\square$

## 8 Transfer maps

In this section we prove Lemma 7.3. For this purpose, we use the transfer maps defined as follows.

**Definition 8.1.** (i) For a connected based space  $(X, x_0)$ , let  $\text{SP}^\infty(X)$  denote the infinite symmetric product of  $X$  defined by  $\text{SP}^\infty(X) = \lim_{d \rightarrow \infty} \text{SP}^d(X) = \bigcup_{d \geq 0} \text{SP}^d(X)$ , where the space  $\text{SP}^d(X)$  is regarded as the subspace of  $\text{SP}^{d+1}(X)$  by identifying  $\sum_{k=1}^d x_k$  with  $\sum_{k=1}^d x_k + x_0$ . So the space  $\text{SP}^\infty(X)$  may be regarded as the abelian monoid generated by  $X$  with the unit  $x_0$ .

(ii) Since there is a homotopy equivalence  $\text{Poly}_n^{(d-1)n,m} \simeq \text{Poly}_n^{dn-1,m}$  (by Theorem 5.17), by using this identification we define the map

$$\tau : \text{Poly}_n^{dn,m} \rightarrow \text{SP}^\infty(\text{Poly}_n^{dn-1,m}) \simeq \text{SP}^\infty(\text{Poly}_n^{(d-1)n,m}) \quad \text{by}$$

$$(f_1(z), \dots, f_m(z)) \mapsto \sum_{1 \leq i_1, \dots, i_m \leq dn} \left( \prod_{k=1, k \neq i_1}^{dn} (z - a_{k,1}), \dots, \prod_{k=1, k \neq i_m}^{dn} (z - a_{k,m}) \right),$$

where  $(f_1(z), \dots, f_m(z)) \in \text{Poly}_n^{dn,m}$  and  $f_j(z) = \prod_{k=1}^{dn} (z - a_{k,j})$  for  $1 \leq j \leq m$ . The map  $\tau$  naturally extends to a homomorphism of abelian monoid

$$(8.1) \quad \tau_{d-1} : \text{SP}^\infty(\text{Poly}_n^{dn,m}) \rightarrow \text{SP}^\infty(\text{Poly}_n^{(d-1)n,m}).$$

For each  $1 \leq k < d$ , define the transfer map

$$(8.2) \quad \tau_{k,d} : \mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m}) \rightarrow \mathrm{SP}^\infty(\mathrm{Poly}_n^{dk,m})$$

as the composite  $\tau_{k,d} = \tau_k \circ \tau_{k+1} \circ \cdots \circ \tau_{d-1}$ , i.e.

$$\tau_{k,d} : \mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m}) \xrightarrow{\tau_{d-1}} \mathrm{SP}^\infty(\mathrm{Poly}_n^{(d-1)n,m}) \rightarrow \cdots \xrightarrow{\tau_k} \mathrm{SP}^\infty(\mathrm{Poly}_n^{kn,m}).$$

In particular, we set  $\tau_{d,d} = \mathrm{id}$  for  $k = d$ .  $\square$

**Lemma 8.2.** (i) *The induced homomorphism  $(s_{d-1,d})_* : H_*(\mathrm{Poly}_n^{(d-1)n,m}, \mathbb{Z}) \rightarrow H_*(\mathrm{Poly}_n^{dn,m}, \mathbb{Z})$  is a monomorphism.*

(ii) *The map*

$$\mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m}) \xrightarrow[\simeq]{(\tilde{p}_d, \tau_{d-1,d})} \mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m} / \mathrm{Poly}_n^{(d-1)n,m}) \times \mathrm{SP}^\infty(\mathrm{Poly}_n^{(d-1)n,m})$$

*is a homotopy equivalence, where*

$$\tilde{p}_k : \mathrm{SP}^\infty(\mathrm{Poly}_n^{kn,m}) \rightarrow \mathrm{SP}^\infty(\mathrm{Poly}_n^{kn,m} / \mathrm{Poly}_n^{(k-1)n,m})$$

*denotes the map induced from the natural projection*

$$\mathrm{Poly}_n^{kn,m} \rightarrow \mathrm{Poly}_n^{kn,m} / \mathrm{Poly}_n^{(k-1)n,m}.$$

*Proof.* It is well-known that there is a natural isomorphism  $\pi_k(\mathrm{SP}^\infty(X)) \cong \tilde{H}_k(X, \mathbb{Z})$  for any connected space  $X$  and any  $k \geq 0$ . Furthermore, note that the equality  $\tilde{p}_k \circ \tau_{k,d-1} = \tilde{p}_k \circ \tau_{k,d} \circ s_{d-1,d}$  (up to homotopy equivalence) holds for each  $1 \leq k < d$ . Thus we can show that

$$(\tau_{k,d})_* \circ (s_{d-1,d})_* \equiv (\tau_{k,d-1})_* \pmod{\mathrm{Im} (s_{k-1,k})_*}$$

on  $H_*(\mathrm{Poly}_n^{kn,m}, \mathbb{Z})$  for each  $1 \leq k < d$ . Then by using [10, Lemma 2], we can prove that  $(s_{d-1,d})_*$  is a monomorphism and that the map  $(\tilde{p}_d, \tau_{d-1,d})$  induces an isomorphism on the homotopy group  $\pi_k(\ )$  for any  $k$ . Hence, the map  $(\tilde{p}_d, \tau_{d-1,d})$  is a homotopy equivalence.  $\square$

If we use the above result, it is easy to prove the following result.

**Corollary 8.3.** *The map*

$$\tilde{\tau}_d = \prod_{k=1}^d \tilde{\tau}_{k,d} : \mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m}) \xrightarrow{\simeq} \prod_{k=1}^d \mathrm{SP}^\infty(\mathrm{Poly}_n^{kn,m} / \mathrm{Poly}_n^{(k-1)n,m})$$

*is a homotopy equivalence, where the map  $\tilde{\tau}_{k,d}$  denotes the composite of maps*

$$\mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m}) \xrightarrow{\tau_{k,d}} \mathrm{SP}^\infty(\mathrm{Poly}_n^{kn,m}) \xrightarrow{\tilde{p}_k} \mathrm{SP}^\infty(\mathrm{Poly}_n^{kn,m} / \mathrm{Poly}_n^{(k-1)n,m}). \quad \square$$

**Definition 8.4.** Let  $N$  be a positive integer and assume that  $1 \leq j < d$ .

(i) Let  $q_{d,j}^{(N)} : F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^d \rightarrow F(\mathbb{C}, d) \times_{S_d} (S^{2N-1})^d$  denote the natural covering projection corresponding the subgroup  $S_j \times S_{d-j} \subset S_d$ . Define the transfer

$$(8.3) \quad \sigma^{(N)} : F(\mathbb{C}, d) \times_{S_d} (S^{2N-1})^d \rightarrow \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^d)$$

for the covering projection  $q_{d,j}^{(N)}$  by

$$\sigma^{(N)}(x) = \sum_{\tilde{x} \in q_{d,j}^{-1}(x)} \tilde{x} \quad \text{for } x \in F(\mathbb{C}, d) \times_{S_d} (S^{2N-1})^d.$$

(ii) Let  $\rho_j^{(N)} : F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^d \rightarrow F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^j$  denote the map onto the first  $j$  coordinates of  $(S^{2N-1})^d$ , and define the map

$$\sigma_j^{(N)} : F(\mathbb{C}, d) \times_{S_d} (S^{2N-1})^d \rightarrow \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^j)$$

by  $\sigma_j^{(N)} = \mathrm{SP}^\infty(\rho_j^{(N)}) \circ \sigma^{(N)}$ . The map  $\sigma_j^{(N)}$  naturally extends to a map

$$(8.4) \quad \tilde{\sigma}_j^{(N)} : \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_d} (S^{2N-1})^d) \rightarrow \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^j)$$

by the usual addition  $\tilde{\sigma}_j^{(N)}(\sum_k x_k) = \sum_k \sigma_j^{(N)}(x_k)$ .

(iii) Let

$$\mathcal{I}'_{j,d} : F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2mn-3})^j \rightarrow \mathrm{Poly}_n^{j^{n,m}}$$

denote the  $C_2$ -structure map given by the similar way as  $\mathcal{I}'_d$  was defined.  $\square$

**Lemma 8.5** ([6]). *Let  $1 \leq j < d$ . Then the stable map*

$$\sigma_j^{(N)} \circ e_{d,N} : D_d(S^{2N-1}) \rightarrow \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2N-1})^j)$$

*is null-homotopic.*

*Proof.* The case  $N = 1$  was proved in [6, page 44-45] and the case  $N \geq 2$  can be proved completely same way.  $\square$

The following is easy to verify:

**Lemma 8.6.** *Let  $m$  and  $n$  be positive integers  $\geq 2$ . Then if  $1 \leq j < d$  and  $N = mn - 1$ , the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_d} (S^{2mn-3})^d) & \xrightarrow{\tilde{\sigma}_j^{(mn-1)}} & \mathrm{SP}^\infty(F(\mathbb{C}, d) \times_{S_j \times S_{d-j}} (S^{2mn-3})^d) \\ \mathrm{SP}^\infty(\mathcal{I}'_d) \downarrow & & \mathrm{SP}^\infty(\mathcal{I}'_{j,d}) \downarrow \\ \mathrm{SP}^\infty(\mathrm{Poly}_n^{dn,m}) & \xrightarrow{\tau_{j,d}} & \mathrm{SP}^\infty(\mathrm{Poly}_n^{jn,m}) \quad \square \end{array}$$



**Lemma 8.7.** *If  $m \geq 2$  and  $n \geq 2$  be positive integers, the stable map*

$$\tau_{d-1,d} \circ \mathrm{SP}^\infty(\mathcal{I}'_d) \circ \mathrm{SP}^\infty(e_d) : \mathrm{SP}^\infty(\Sigma^{2(mn-2)d} D_d) \rightarrow \mathrm{SP}^\infty(\mathrm{Poly}_n^{(d-1)n,m})$$

*is null-homotopic.*

*Proof.* Note that  $\sigma_{d-1}^{(mn-1)} \circ e_d$  is null-homotopic by Lemma 8.5 (cf. (6.7)). By Lemma 8.6 we see that

$$\begin{aligned} \tau_{d-1,d} \circ \mathrm{SP}^\infty(\mathcal{I}'_d) \circ \mathrm{SP}^\infty(e_d) &= \mathrm{SP}^\infty(\mathcal{I}'_{d-1,d}) \circ \tilde{\sigma}_{d-1}^{(mn-1)} \circ \mathrm{SP}^\infty(e_d) \\ &= \mathrm{SP}^\infty(\mathcal{I}'_{d-1,d}) \circ \mathrm{SP}^\infty(\sigma_{d-1}^{(mn-1)} \circ e_d) \simeq *. \end{aligned}$$

Thus the map  $\tau_{d-1,d} \circ \mathrm{SP}^\infty(\mathcal{I}'_d) \circ \mathrm{SP}^\infty(e_d)$  is null-homotopic.  $\square$

Now it is ready to prove Lemma 7.3.

*Proof of Lemma 7.3.* Since the first assertion follows from (i) of Lemma 8.2, it suffices to prove the assertion (ii). First, it follows from Theorem 7.1 that the homomorphism  $(\mathcal{I}'_d \circ e_d)_* : H_*(\Sigma^{2(mn-2)d} D_d, \mathbb{Z}) \rightarrow H_*(\mathrm{Poly}_n^{dn,m}, \mathbb{Z})$  is a monomorphism. Next, by (ii) of Lemma 8.2 the homomorphism

$$H_*(\mathrm{Poly}_n^{dn,m}) \xrightarrow[\cong]{(\tilde{p}_d, \tau_{d-1,d})_*} H_*(\mathrm{Poly}_n^{dn,m} / \mathrm{Poly}_n^{(d-1)n,m}) \oplus H_*(\mathrm{Poly}_n^{(d-1)n,m})$$

is an isomorphism. Hence, the composite  $((\tilde{p}_d)_* \circ (\mathcal{I}'_d \circ e_d)_*, (\tau_{d-1,d})_* \circ (\mathcal{I}'_d \circ e_d)_*)$

$$H_*(\Sigma^{2(mn-2)d} D_d, \mathbb{Z}) \rightarrow H_*(\mathrm{Poly}_n^{dn,m} / \mathrm{Poly}_n^{(d-1)n,m}, \mathbb{Z}) \oplus H_*(\mathrm{Poly}_n^{(d-1)n,m}, \mathbb{Z})$$

is a monomorphism. However, by Lemma 8.7,  $(\tau_{d-1,d})_* \circ (\mathcal{I}'_d \circ e_d)_*$  is trivial. Thus, the homomorphism

$$(\tilde{p}_d)_* \circ (\mathcal{I}'_d \circ e_d)_* : H_*(\Sigma^{2(mn-2)d} D_d, \mathbb{Z}) \rightarrow H_*(\mathrm{Poly}_n^{dn,m} / \mathrm{Poly}_n^{(d-1)n,m}, \mathbb{Z})$$

is a monomorphism. Since  $\Psi_d = \tilde{p}_d \circ \mathcal{I}'_d \circ e_d$ , the homomorphism

$$(\Psi_d)_* : H_*(\Sigma^{2(mn-2)d} D_d, \mathbb{Z}) \rightarrow H_*(\mathrm{Poly}_n^{dn,m} / \mathrm{Poly}_n^{(d-1)n,m}, \mathbb{Z})$$

is a monomorphism.  $\square$

**Funding.** The second author was supported by JSPS KAKENHI Grant Number 26400083, Japan.

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